Extremal Problems for Graphs and Hypergraphs

by

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Thesis

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Dedicated to my family and friends, whose support crystallized into this work.

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Contents

1 INTRODUCTION **1**

1.1 The Erdős-Komlós Function 2

1.2 The generalized Turán problem for counting K_3's 5

1.2.1 Complete bipartite graphs 7

1.2.2 Even cycles 7

1.2.3 Paths 8

1.3 Turán numbers of 3-graphs 8

1.4 Organization and Contribution of authors 11

2 MAXIMUM SUBGRAPHS WITHOUT FIXED HYPERGRAPH FAMILIES **13**

2.1 Background 13

2.2 Our Results 14

2.2.1 Constant Sequences 14

2.2.2 Non-constant Sequences 15

2.3 Proofs of Theorems 2.2.1 and 2.2.3 20

2.4 Proof of Theorem 2.2.7 22

2.5 Proof of Theorem 2.2.8 31

2.6 Proof of Theorem 2.2.10 35

2.7 Further Problems and Discussion 38

3 THE GENERALIZED TURÁN PROBLEM FOR COUNTING TRIANGLES 41

3.1 Background 41

3.2 Our Results 42

3.3 Preliminaries 44

3.4 Suspension of complete bipartite graphs 46

3.5 Suspension of even cycles 48

3.6 Suspension of paths 52

3.6.1 Proof of Proposition 3.2.3 53

3.6.2 Proof of Proposition 3.2.5 54

3.6.3 Proof of Theorem 3.2.4 56

4 EXTREMAL NUMBERS OF HYPERGRAPH SUSPENSIONS OF EVEN CYCLES 71

4.1 Background 71

4.2 Our Results 72

4.2.1 3-Graphs 72

4.2.2 Graphs 74

4.3 Lower bounds on $\mathrm{ex}(n, \widetilde{C}_{2k})$ 76

4.3.1 The 3-graphs $D_3(k, q)$ 78

4.3.2 The bipartite graphs $D'(k, q)$ 85

4.3.3 Proof of Theorem 4.2.2 89

4.4 The arc construction and Wenger's construction 89

4.4.1 Proof of Proposition 4.2.4 90

4.4.2 Proof of Theorem 4.2.5 91

5 VITA **101**

A BINOMIAL IDENTITIES USED IN CHAPTER 2 **103**

B ADDITIONAL DETAILS FOR CHAPTER 3 **107**

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List of Figures

2-1 An example: $H(1,2,3) \in \mathbf{EIP}_3$ 16

2-2 Theorems 2.2.7 and 2.2.8 for $\vec{\mathbf{b}} = (b_1, b_2, 1)$ 18

2-3 $\vec{\mathbf{b}} = (b_1, b_2, 1)$ 19

2-4 Constructing F_m from copies of K 29

3-1 (left): third triangle on ax_1, (right): third triangle on ab 58

3-2 The graphs K_5, K_5^-, K_4, $K_{2,2,2}$, $Q_{3,2}$ and $K_{1,2,2}$. 60

3-3 (left): K_5^-'s are edge-disjoint; (right): K_4's are edge-disjoint. 61

3-4 $W_{4,2}$ is a block by itself. 62

3-5 $Q_{3,2}$ is a block by itself. 62

3-6 The different ways two induced W_4's can intersect. 63

3-7 xu and yu cannot both be heavy. 65

3-8 Codegree of $uv \in E(H)$ is at most 2. 65

3-9 K_6^- is a block by itself. 67

3-10 Graphs to be cleaned from H. 68

3-11 abc, abx, acy are triangles with $x \neq y$. 69

3-12 Edge xa can be light or heavy. 70

B-1 All edges but ac cannot lie in external triangles. 108

B-2 acx is a triangle. 109

B-3 Edge ad lies in triangle adx, $x \notin \{a, b, c, d, e, f\}$ 110

B-4 adx is an external triangle, ab, bc, ca are non-edges in H. 110

B-5 bdx and ady are triangles, (left: $x = y$, right: $x \neq y$) 112

B-6 K_5^- is a block by itself. 113

B-7 Case 1: $y = z$, aey and cdy are triangles in H. 114

B-8 Case 2: $y \neq z$, aey and cdz are triangles in H. 115

B-9 xa cannot lie in an external triangle xay. 116

List of Tables

4.1 Automorphisms of $D(q)$ 80

4.2 The isomorphism $D'(6,q) \to D(6,q)$ 87

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SUMMARY

A hypergraph H consists of vertex set $V(H)$ and an edge set $E(H)$ which is a collection of subsets of $V(H)$. When the edges of H have size 2, H is said to be a graph. For a fixed graph G, the size of the largest n-vertex graph containing no copy of G is a widely studied problem in combinatorics. It was first introduced by Turán [68], and has since then been studied extensively by several authors. In our work, we mainly study the extension of Turán's problem in the following three directions:

1. **The Erdős-Komlós Function.** For a family of hypergraphs $\mathcal{H}$, a hypergraph F is said to be $\mathcal{H}$-free if it does not contain copies of H for every $H \in \mathcal{H}$. Let $f(m, \mathcal{H})$ denote the size of the largest $\mathcal{H}$-free subgraph guaranteed to exist in every hypergraph on m edges. This function was first introduced by Erdős and Komlós [24] in the context of union-free families, and various other special cases have been extensively studied since then. We attempt to develop a general theory for all these problems, and ask the following question: for which sequence of hypergraph families $\{\mathcal{H}_m\}_{m=1}^{\infty}$ is $f(m, \mathcal{H}_m)$ bounded (by a constant) as $m \to \infty$? We consider restrictions of this question in special cases, and prove several general bounds that answer the question in certain situations. The problem seems to be hopeless to solve in its full generality.

2. **The generalized Turán problem for counting K_3's.** The graph K_3, or the triangle, consists of vertices a, b, c and edges ab, bc, ca. Let $\mathrm{ex}(n, K_3, H)$ denote the maximum number of triangles in any n-vertex graph that does not contain a copy of H as a subgraph. The systematic study of $\mathrm{ex}(n, K_3, H)$ was

initiated by Alon and Shikhelman in 2016 [1]. In particular, they determined the asymptotics of $\mathrm{ex}(n, K_3, H)$ when the chromatic number of H is at least 4, and obtained several bounds for bipartite graphs H. We consider this problem when H has chromatic number 3, and focus our attention on a simple class of 3-chromatic graphs. The suspension $\widehat{H}$ of a graph H is obtained from H by adding a new vertex adjacent to all vertices of H. We obtain bounds on $\mathrm{ex}(n, K_3, \widehat{H})$ when H is a path, even cycle, or complete bipartite graph.

3. **Turán numbers of 3-graphs.** For fixed $k \geq 2$, determining the order of magnitude of the number of edges in an n-vertex graph not containing C_{2k}, the cycle of length $2k$, is a long-standing open problem. It was shown in 1974 by Bondy and Simonovits [9] that this number is bounded above by $20k \cdot n^{1+1/k}$, and several authors have improved the constant multiple from $20k$ down to $80\sqrt{k}\log k$ in recent years. However, lower bounds having order of magnitude $n^{1+1/k}$ have only been obtained for $k \in \{2, 3, 5\}$. We consider an extension of this problem to triple systems. Given a vertex x in a hypergraph H, the link of x in H is the graph obtained by joining two vertices of $V(H)$ if they form an edge in H together with x. We prove that the number of triples in an n-vertex triple system which does not contain a C_6 in the link of any vertex, has order of magnitude $n^{7/3}$. We also prove that for C_8, the corresponding number has order of magnitude at least $(n^{11/5})$. Additionally, using a result of Mellinger and Mubayi [53, 54] we construct new families of dense C_6-free bipartite graphs with n vertices and number of edges having order of magnitude of $n^{4/3}$.

Chapter 1

INTRODUCTION

A simple, undirected graph $H = (V(H), E(H))$ consists of a vertex set $V(H)$ and edge set $E(H) \subseteq \binom{V(H)}{2}$. In this document, unless mentioned otherwise, all graphs are simple and undirected. A hypergraph H on vertex set $V(H)$ is a subset of its power set, which we denote by $2^{V(H)}$. Given a graph H, determining the maximum number of edges in an n-vertex graph which does not contain H as a subgraph is a central problem in extremal graph theory. This number is denoted by $\mathrm{ex}(n, H)$, and is known as the *extremal number* or the *Turán number* of the graph H. The problem was first introduced and studied by Turán in 1941 [68], where he determined $\mathrm{ex}(n, K_t)$ for $t \geq 2$, where K_t is the complete graph on t vertices. The number $\mathrm{ex}(n, H)$ has since then been extensively studied for different graphs H by various authors in subsequent years, leading to the emergence of the field of extremal graph theory.

As determining the exact values of $\mathrm{ex}(n, H)$ turns out to be notoriously difficult for general graphs H, researchers have invested effort in determining the asymptotic behavior of $\mathrm{ex}(n, H)$ as n becomes large and H is fixed. Before presenting the main results of this thesis, we shall introduce some asymptotic notation. Given two functions $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ if $\lim\limits_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| \leq c$ for some fixed $c > 0$. Moreover, if $\lim\limits_{n\to\infty} \left|\frac{f(n)}{g(n)}\right| = 0$, then we say that $f(n) = o(g(n))$ or $g(n) = \omega(f(n))$.

Perhaps the most seminal result in the area of extremal graph theory is that of Erdős and Stone [19], which determines the asymptotic behavior of $\mathrm{ex}(n, H)$ for all graphs H with chromatic number $\chi(H)$ at least 3. More precisely, it was shown that when $\chi(H) \geq 3$,

$$\mathrm{ex}(n, H) = \left(\frac{\chi(H) - 2}{\chi(H) - 1} + o(1)\right)\binom{n}{2}. \tag{1.1}$$

Unfortunately, the problem of determining even the asymptotic behavior of $\mathrm{ex}(n, H)$ when $\chi(H) = 2$, i.e. when H is bipartite, is still wide open in general, and is a rich and active area of research. For a comprehensive survey on the history of Turán type problems, the reader is referred to the monograph by Simonovits [66].

Our work is mainly focused on extending Turán's question in three directions.

1.1 The Erdős-Komlós Function

The first question that we consider in this thesis, is related to families of hypergraphs. For two hypergraphs H and H', H' is said to be a *subgraph* of H if $H' \subseteq H$. Additionally, two hypergraphs H and H' are said to be isomorphic (denoted as $H \cong H'$) if there is a bijection of sets from $V(H)$ to $V(H')$ that maps edges in H to edges in H', and non-edges in H to non-edges in H'.

For a fixed hypergraph H, we say that a hypergraph F is H-free if there is no $F' \subseteq F$ such that F' and H are isomorphic. Given a family of hypergraphs $\mathcal{H}$, we say F is $\mathcal{H}$-free if and only if F is H-free for every $H \in \mathcal{H}$. Let $f(m, \mathcal{H})$ denote the size of the largest $\mathcal{H}$-free subgraph guaranteed to exist in every hypergraph on m edges. The size of the largest $\mathcal{H}$-free subgraph in F is denoted by $\mathrm{ex}(F, \mathcal{H})$, and thus,

$$f(m, \mathcal{H}) = \min_{|F|=m} \mathrm{ex}(F, \mathcal{H}).$$

When the family $\mathcal{H}$ consists of a single hypergraph H, we abuse notation and write $f(m, H)$ instead of $f(m, \{H\})$. Observe that $f(m, \mathcal{H}) \geq c$ means that every F with

m edges contains an $\mathcal{H}$-free subgraph $F' \subseteq F$ with $|F'| = c$.

The function $f(m, \mathcal{H})$ was introduced by Erdős and Komlós in 1969 [24] in the context of union-free families. They proved that if $\mathcal{U}_2$ denotes the (infinite) family of hypergraphs $\{A, B, C\}$ with $A \cup B = C$,

$$\sqrt{m} \leq f(m, \mathcal{U}_2) \leq \sqrt{8m}.$$

These bounds were further improved by Kleitman [43], and later by Erdős and Shelah [27] to $\sqrt{2m} \leq f(m, \mathcal{U}_2) \leq 2\sqrt{m}$. Finally, Fox, Lee and Sudakov [34] brought closure to this problem in 2012 via proving the exact equality

$$f(m, \mathcal{U}_2) = \left\lfloor \sqrt{4m+1} \right\rfloor - 1.$$

Erdős and Komlós [24] also studied the function $f(m, \mathcal{B}_2)$ when $\mathcal{B}_2$ is the family of hypergraphs $\{A, B, C, D\}$ with $A \cup B = C$ and $A \cap B = D$. In particular, it was shown that $f(m, \mathcal{B}_2) \leq \frac{3}{2} m^{2/3}$. They also conjectured that this bound is tight, which was later settled by Barát, Füredi, Kantor, Kim and Patkós [3]. These authors also considered $f(m, \mathcal{U}_a)$ for the family $\mathcal{U}_a = \{\{A_1, \ldots, A_a, B\} : B = A_1 \cup \cdots \cup A_a\}$ as a generalization of Erdős and Komlós's question. The function $f(m, \mathcal{U}_a)$ was studied in greater detail by Fox, Lee and Sudakov [34].

The same problem has been studied in the special case when $\mathcal{H}$ is a family of graphs. Let $K_{a,b}, K_{a,b,c}, C_k$ and P_k denote the complete bipartite graph with parts of size a and b, the complete tripartite graphs with parts of size a, b and c, the cycle on k vertices, and the path with k edges, respectively. Let $f_2(m, \mathcal{H})$ denote the maximum size of a $\mathcal{H}$-free subgraph in all graphs with m edges. Erdős and Bollobás [61] asked whether $f_2(m, C_4) \geq c \cdot m^{3/4}$ in a workshop in 1966. Later, Erdős remarked in [21] that the answer is likely $\Theta(m^{2/3})$. Foucaud, Krivelevich and Perarnau [32] prove this conjecture in a more general form, by proving that

$$f_2(m, \{C_4, \ldots, C_{2r}\}) = O(m^{(r+1)/(2r)}).$$

They also show that this bound is tight upto logarithmic factors, and in particular,

$$f_2(m, C_4) \geq \Omega\left(\frac{m^{2/3}}{\log m}\right). \tag{1.2}$$

Further, using (1.1), they observe that if $\mathcal{H}$ contains no bipartite graphs, then the function $f_2(m, \mathcal{H})$ is linear in m.

In a different direction, Conlon, Fox and Sudakov consider $f_2(m, K_{r,r})$ [14, 15]. They prove that

$$f_2(m, K_{r,r}) \geq \frac{1}{4} m^{\frac{r}{r+1}}.$$

Note that since $K_{2,2} = C_4$, this improves (1.2) by a logarithmic factor via proving $f_2(m, C_4) \geq \Omega(m^{2/3})$. They also extend the problem to the case of hypergraphs.

In our work, we attempt to obtain a general theory for these problems. Recall that a sequence is bounded if all its terms are bounded by a fixed constant. We ask the following question:

$$\begin{gathered}\text{For which sequence of families } \{\mathcal{H}_m\}_{m=1}^{\infty} \\ \text{is } f(m, \mathcal{H}_m) \text{ bounded (as } m \to \infty\text{)?}\end{gathered} \tag{1.3}$$

As Question (1.3) turns out to be very difficult to answer, together with Mubayi, we consider $f(m, \mathcal{H}_m)$ for specific sequences of hypergraph families $\mathcal{H}_m$ in Chapter 2.

In particular, when $\mathcal{H}_m = \mathcal{H}$ for every m, i.e. when $\{\mathcal{H}_m\}_{m=1}^{\infty}$ is the constant sequence and when $\mathcal{H}$ consists of only finitely many elements, we answer Question (1.3) completely as follows. A q-sunflower is a hypergraph $\{A_1, \ldots, A_q\}$ such that $A_i \cap A_j = \bigcap_{s=1}^{q} A_s$ for every $i \neq j$. This common intersection is referred to as the *core* of the sunflower.

Theorem (2.2.1)**.** For a family of hypergraphs $\mathcal{H}$ with finitely many members, if $\mathcal{H}$ contains a q-sunflower with sets of equal size, then $f(m, \mathcal{H}) \leq q - 1$. Otherwise, $f(m, \mathcal{H}) \to \infty$ as $m \to \infty$.

For nonconstant sequences $\{\mathcal{H}_m\}$, we specialize to the case where each $\mathcal{H}_m$ is singleton, i.e., $\mathcal{H}_m = \{H_m\}$ for some hypergraph H_m, $m \geq 1$, and further restrict ourselves to the case when each H_m has equal number of edges. In other words, we study the following question:

$$\begin{gathered}\text{For which sequences of } k\text{-edge hypergraphs } \{H_m\}_{m=1}^{\infty} \\ \text{is } f(m, H_m) \text{ bounded (as } m \to \infty\text{)?}\end{gathered} \tag{1.4}$$

In particular, we observe that for $f(m, H_m)$ to be bounded, all but finitely many of the H_m's needs to satisfy a property which we call the *equal intersection property*. A hypergraph H satisfies the *equal intersection property* if all its edges have the same size, and any i of its edges intersect in the same number of edges, for every $i \geq 2$. We further narrow down hypergraph sequences $\{H_m\}$ for which $f(m, H_m)$ is bounded depending on relations between the intersection sizes. This case is covered in great detail in Chapter 2, specifically, Theorems 2.2.7, 2.2.8 and 2.2.10.

1.2 The generalized Turán problem for counting K_3's

The second direction that we investigate considers the so-called *generalized Turán numbers*. For graphs T and H with no isolated vertices and integer n, the generalized Turán number $\mathrm{ex}(n, T, H)$ is the largest number of copies of T in an H-free n-vertex graph. When $T = K_2$, this is the Turán number $\mathrm{ex}(n, H)$ of the graph H. The systematic study of $\mathrm{ex}(n, T, H)$ for $T \neq K_2$ was initiated by Alon and Shikhelman [1]. Previously, there had been sporadic results determining $\mathrm{ex}(n, T, H)$ for various T and H. Possibly the first authors that considered this general version were Erdős [18] and Bollobas [7], who analyzed $\mathrm{ex}(n, K_t, K_r)$ for $t < r$. Several cases for $H = K_r$ were later studied by Györi in [41].

More recently, $\mathrm{ex}(n, K_3, C_5)$ has received considerable attention in [8, 1]. The best known bounds on $\mathrm{ex}(n, K_3, C_5)$ are given by Ergemlidze, Methuku, Salia and

Györi [31], where they prove that

$$\left(\frac{1}{3\sqrt{3}}o(1)\right)n^{3/2} \leq \mathrm{ex}(n, K_3, C_5) \leq \left(\frac{1}{2\sqrt{2}} + o(1)\right)n^{3/2}.$$

The same problem when reversing the roles of K_3 and C_5 was introduced by Erdős in 1984 [29]. It was conjectured that $\mathrm{ex}(n, C_5, K_3) \leq \left(\frac{n}{5}\right)^5$, whence the bound was known to be attained for $5 \mid n$. Györi [39] proved that $\mathrm{ex}(n, C_5, K_3) \leq c\left(\frac{n+1}{n}\right)^5$, where $c \approx 1.030$. Erdős' conjecture was recently proved by Grzesik using the technique of flag algebras [38].

On the other hand, Györi and Li also prove several upper and lower bounds on $\mathrm{ex}(n, K_3, C_{2k+1})$ in [40]. In a similar spirit, Luo [50] bounded the number of s-cliques in graphs without cycles of length at least k.

Alon and Shikhelman characterized graphs T, H with $\mathrm{ex}(n, T, H) = \Theta(n^{|V(T)|})$ in [1]. They also studied the problem for several other choices of T and H, such as when T is a bipartite graph or a tree, and H is a tree. When $T = K_3$, they gave a characterization of all H for which $\mathrm{ex}(n, K_3, H) = \Theta(n)$. Improving a result in [1], Ma and Qiu more precisely determine the asymptotics of $\mathrm{ex}(n, K_m, H)$ for general graphs H with $\chi(H) > m$ in [51].

On the other hand, in [36], Gerbner and Palmer prove that for graphs T and H with $\chi(H) = k$,

$$\mathrm{ex}(n, T, H) \leq \mathrm{ex}(n, T, K_k) + o(n^{|V(T)|})$$

In the same article they also discuss $\mathrm{ex}(n, P_k, K_{2,t})$ and $\mathrm{ex}(n, C_k, K_{2,t})$.

Our contribution is focused on the special case of $T = K_3$. In this case, [1, 51] determine the asymptotics of $\mathrm{ex}(n, K_3, H)$ when $\chi(H) \geq 3$, and [1] contains various results for $\chi(H) = 2$. Therefore, we further narrow down our focus to graphs H for which $\chi(H) = 3$.

The family of 3-chromatic graphs that we study in Chapter 3 are the so-called *sus-*

pensions of bipartite graphs. For any graph H, let $\widehat{H}$ denote the suspension $K_1 \vee H$ obtained by adding a new vertex adjacent to every vertex of H. In joint work with Mubayi, we obtain upper and lower bounds on $\mathrm{ex}(n, K_3, \widehat{H})$ for $H \in \{K_{a,b}, C_{2k}, P_k\}$, where P_k denotes the path on k edges.

1.2.1 Complete bipartite graphs

For $H = K_{a,b}$, we prove the following result:

Theorem (3.2.1)**.** For fixed $1 \le a \le b$ and $n \to \infty$,

$$\mathrm{ex}(n, K_3, K_{1,a,b}) = o(n^{3-\frac{1}{a}}). \tag{1.5}$$

Notice that setting $a = b = 2$ in (3.2) yields $\mathrm{ex}(n, K_3, K_{1,2,2}) = o(n^{5/2})$, where $K_{1,2,2} = \widehat{C}_4$ is the 4-wheel graph. This is related to a question of Mubayi and Verstraëte [58], where the authors asked whether $\mathrm{ex}(n, K_3, K_{1,2,2}) = O(n^2)$. Although the answer to this question is still unknown, our results give

$$\Omega(n^2) \le \mathrm{ex}(n, K_3, K_{1,2,2}) < o(n^{5/2}).$$

Narrowing the (huge) gap in the bounds above is perhaps the most attractive open problem in this area.

1.2.2 Even cycles

We also analyze the case $H = C_{2k}$, and note that there is a large gap in the upper and lower bounds for $\mathrm{ex}(n, K_3, \widehat{C}_{2k})$.

Theorem (3.2.2)**.** For fixed $k \ge 2$ and $n \to \infty$,

$$\Omega(n^2) \le \mathrm{ex}(n, K_3, \widehat{C}_{2k}) < o(n^{2+\frac{1}{k}}). \tag{1.6}$$

1.2.3 Paths

When $H = P_k$, we give tight bounds on $\mathrm{ex}(n, K_3, P_k)$ for $k = 3, 4, 5$.

For $n \geq k \geq 3$, we first observe (Proposition 3.2.3) that,

$$\left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8} \leq \mathrm{ex}(n, K_3, \widehat{P}_k) \leq \frac{k-1}{12} \cdot n^2 + \frac{(k-1)^2}{12} \cdot n, \tag{1.7}$$

where the lower bound holds when n is a multiple of $4\lfloor \frac{k-1}{2} \rfloor$.

We believe that the lower bound above is asymptotically tight for all fixed $k \geq 3$ and prove this for the first three cases $k = 3$, 4 and 5.

Theorem (3.2.4)**.** For $k = 3$, 4 and 5,

$$\mathrm{ex}(n, K_3, \widehat{P}_k) = \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8} + o(n^2). \tag{1.8}$$

When $k = 3$ or $k = 5$, the error term can be improved to $O(n)$.

1.3 Turán numbers of 3-graphs

Given $r \geq 2$, an r-uniform hypergraph, or simply an r-graph H on vertex set $V(H)$ is a subset of $\binom{V(H)}{r}$, the set of all subsets of $V(H)$ of size r. The direct analogue of the Turán problem for r-graphs asks the following question: what is the largest size of an r-graph on n vertices that does not contain a copy of H as a subgraph? This number is known as the Turán number or the extremal number of H, and is denoted by $\mathrm{ex}_r(n, H)$. Several lower and upper bounds on $\mathrm{ex}_r(n, H)$ (for different values of r and H) have been obtained since 1941 when Turán first introduced $\mathrm{ex}(n, H)$. For a survey on the hypergraph Turán problem, we refer the reader to [42].

Let us now introduce one of the most basic open questions in extremal graph theory, which is to determine the asymptotic behavior of $\mathrm{ex}(n, C_{2k})$ for all $k \geq 2$. The first bound on this problem was due to Erdős [30], who showed that $\mathrm{ex}(n, C_4) \leq O(n^{3/2})$.

In the later years, due to work by Kövari, Sós and Turán [44], Erdős and Rényi [26] and Brown [11], it is now known that

$$\mathrm{ex}(n, C_4) = \left(\frac{1}{2} + o(1)\right) n^{3/2}.$$

Afterwards, it was conjectured by Erdős and Simonovits [28] that $\mathrm{ex}(n, C_{2k}) = (\frac{1}{2} + o(1))n^{1+1/k}$ for all $k \geq 2$. However, the value of the constant is not known for no other cycles longer than C_4. In particular, their conjecture is false even for C_6, as demonstrated by Füredi, Naor and Verstraëte [35]:

$$0.5338n^{4/3} < \mathrm{ex}(n, C_6) \leq 0.6272n^{4/3}$$

For general $k \geq 2$, the asymptotically best known upper bound on $\mathrm{ex}(n, C_{2k})$ was given by Bondy and Simonovits in 1974 [10], who proved that $\mathrm{ex}(n, C_{2k}) \leq 20k \cdot n^{1+1/k}$. In the past several decades, there have been improvements in the constant term due to Verstraëte [69], Lam and Verstraëte [45], Pikhurko [62], and most recently, Bukh and Jiang [12], who provide the current best known upper bound of

$$\mathrm{ex}(n, C_{2k}) \leq 80\sqrt{k} \log k \cdot n^{1+1/k} + O(n).$$

A major open problem for even cycles is to construct C_{2k}-free graphs on n vertices with $\Omega(n^{1+1/k})$ edges. There have been several bipartite constructions based on finite geometries including [63, 11, 5, 52, 72, 49] that have sequentially improved the bounds; however, they give the asymptotically tight bound of $\Omega(n^{1+1/k})$ only for $k \in \{2, 3, 5\}$. For $k \notin \{2, 3, 5\}$, the best known lower bounds are given by the bipartite graphs $CD(k, q)$ [47, 48] for integers $k \geq 2$ and prime powers q. In particular, it is shown that $CD(k, q)$ has at most $2q^{k-t-1}$ vertices and is q-regular where $t = \left\lfloor \frac{k+2}{4} \right\rfloor$. This proves that,

$$\mathrm{ex}(n, C_{2k}) \geq \Omega\left(n^{1+\frac{2}{3k-3+\epsilon}}\right),$$

where $\epsilon = 1$ if k is even and 0 otherwise. The graphs $CD(k,q)$ arise from Lie algebraic incidence structures that approximate the behavior of generalized polygons, and are analyzed in detail in [73].

For a recent survey on the even cycle problem, the reader is referred to [71].

In Chapter 4, we extend the cycle problem to 3-graphs. We consider a specific class of 3-graphs which we call *hypergraph suspensions*. Let H be a 3-graph and $x \in V(H)$ be any vertex of H. The link of x in H, denoted by $L_{x,H}$, is the graph with vertex set $V(H) \setminus \{x\}$ and edge set $\{uv : \{x,u,v\} \in H\}$. For a graph G, the hypergraph suspension $\widetilde{G}$ is a 3-graph defined as follows: add a new vertex x to $V(G)$, and let $\widetilde{G} = \{e \cup \{x\} : e \in E(H)\}$. By definition, $L_{x,\widetilde{G}} = G$.

We analyze the Turán numbers of the 3-graphs $\widetilde{C}_{2k}$ for $k \geq 2$. For $k = 2$, we note that $\widetilde{C}_4$ is the complete 3-partite 3-graph $K^{(3)}_{1,2,2}$, and its extremal number was determined to be $\Theta(n^{5/2})$ by Mubayi in [55]. Hence, we focus our attention on $k \geq 3$.

A simple upper bound on $\mathrm{ex}_3(n, \widetilde{C}_{2k})$ can be obtained via noting that a 3-graph H does not contain $\widetilde{C}_{2k}$ iff $L_{x,H}$ is C_{2k}-free for every $x \in V(H)$, which, together with the classical result of Bondy-Simonovits [9], gives us

$$\mathrm{ex}_3(n, \widetilde{C}_{2k}) \leq n \cdot \mathrm{ex}(n, C_{2k}) \leq O(n^{2+1/k}).$$

On the other hand, a probabilistic argument (Proposition 4.2.1) tells us that $\mathrm{ex}_3(n, \widetilde{C}_{2k}) \geq \Omega(n^{2+1/(2k-1)})$. Our contribution closes this gap for $k = 3$, and narrows it down for $k = 4$. In particular, we prove that

Theorem (4.2.2)**.** For large n,

$$\mathrm{ex}_3(n, \widetilde{C}_6) = \Theta(n^{7/3}) \text{ and } \mathrm{ex}_3(n, \widetilde{C}_8) \geq \Omega(n^{11/5}).$$

In the realm of graphs, we use a construction from Mubayi and Mellinger [53, 54] to produce a new family of C_6-free graphs with n vertices and $(n/2)^{4/3}$ edges for infinitely many values of n, proving that $\mathrm{ex}(n, C_6) \geq 0.3968n^{4/3}$.

Theorem (4.2.5)**.** Let $r \geq 1$, $q = 2^r$, and $\mathbb{F}_q$ denote the finite field of q elements. Suppose $1 \leq s \leq r$ is such that r and s are coprime. Let $G(2^r, s)$ denote the bipartite graph with parts $A = B = \mathbb{F}_q^3$ such that $(a_1, a_2, a_3) \in A$ and $(b_1, b_2, b_3) \in B$ are adjacent iff

$$b_2 + a_2 = b_1 a_1 \text{ and } b_3 + a_3 = a_1 b_1^{2^e}.$$

Then, $G(2^r, s)$ is C_6-free. By construction, it has $2q^3$ vertices and q^4 edges.

Further details on these results and their proofs are discussed in Chapter 4.

1.4 Organization and Contribution of authors

This document is organized in the following manner. In Chapter 2, we consider the Erdős-Komlós function for different hypergraph families. All results in this chapter were obtained jointly with the author's advisor, Dhruv Mubayi. Majority of the results presented in this chapter were submitted and accepted at the *Journal of Combinatorics* [56]. In Chapter 3, we focus our attention on counting triangles in graphs without bipartite suspensions. Most of the results in this chapter were submitted as a paper [57], and the entire work was jointly done with Dhruv Mubayi. Chapter 4 presents the author's work on the hypergraph Turán problem for the 3-graphs $\widetilde{C}_{2k}$, and is available as a preprint [59].

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Chapter 2

MAXIMUM SUBGRAPHS WITHOUT FIXED HYPERGRAPH FAMILIES

2.1 Background

In this chapter, we study the function $f(m, \mathcal{H})$ for different hypergraph families $\mathcal{H}$, which denotes the largest size of an $\mathcal{H}$-free subgraph guaranteed to exist in any hypergraph on m edges. Recall that proving a lower bound $f(m, \mathcal{H}) \geq c$ is equivalent to demonstrating that every hypergraph F on m edges contains an $\mathcal{H}$-free subgraph $F' \subseteq F$ with $|F'| = c$.

This function was introduced by Erdős and Komlós in 1969 [24], who considered the family $\mathcal{U}_2 = \{\{A, B, C\} : A \cup B = C\}$. After progress on $f(m, \mathcal{U}_2)$ in [43] and [27], it was determined exactly by Fox, Lee and Sudakov [34]. Erdős and Shelah [27] also considered $f(m, \mathcal{B}_2)$ for the family $\mathcal{B}_2 = \{\{A, B, C, D\} : A \cup B = C, A \cap B = D\}$, which was determined asymptotically by Barát, Füredi, Kantor, Kim and Patkós [4]. These authors also considered more general problems (see [34] for further work). The same quantity has been studied in the special case when $\mathcal{H}$ is a family of graphs (see, for e.g., [61, 21, 32, 14, 15]).

In the hope of obtaining a general theory for these problems, we consider the following basic question:

$$\text{For which sequence of families } \{\mathcal{H}_m\}_{m=1}^{\infty} \text{ is } f(m, \mathcal{H}_m) \text{ bounded (as } m \to \infty)? \tag{2.1}$$

As question (2.1) is too general to solve completely, we focus our attention on special cases. In subsection 2.1 we state our results for constant $\{\mathcal{H}_m\}_{m=1}^{\infty}$, and in subsection 2.2 we consider non-constant $\{\mathcal{H}_m\}_{m=1}^{\infty}$.

2.2 Our Results

2.2.1 Constant Sequences

Suppose $\{\mathcal{H}_m\}_{m=1}^{\infty}$ is a sequence such that $\mathcal{H}_m = \mathcal{H}$ for every m. First, we note that if $\mathcal{H}$ consists of finitely many members, then the answer to Question (2.1) is given by the following characterization.

Theorem 2.2.1. *Fix a family of hypergraphs $\mathcal{H}$ with finitely many members. If $\mathcal{H}$ contains a q-sunflower with sets of equal size, then $f(m, \mathcal{H}) \leq q - 1$. Otherwise, $f(m, \mathcal{H}) \to \infty$ as $m \to \infty$.*

Next, in the same spirit as the properties of being union-free and having no $\mathcal{B}_2$, if the (infinite) family $\mathcal{H}$ specifies the intersection type of k sets (i.e. whether they are empty or not), then a characterization can be obtained in the form of Theorem 2.2.3. Before stating the theorem, we first define what we call an ℓ-even hypergraph and an ℓ-uneven hypergraph. A k-edge hypergraph is a hypergraph with k edges.

Definition 2.2.2 (ℓ-even and ℓ-uneven hypergraphs)**.** A k-edge hypergraph $H =$

$\{A_1, \ldots, A_k\}$ is said to be *ℓ-even* for some $1 \leq \ell \leq k$ if for every subset $I \subseteq [k]$,

$$\bigcap_{i \in I} A_i \neq \varnothing \text{ iff } |I| \leq \ell.$$

It is said to be *ℓ-uneven* if there exist $I, J \in \binom{[k]}{\ell}$ such that

$$\bigcap_{i \in I} A_i \neq \varnothing \text{ but } \bigcap_{j \in J} A_j = \varnothing.$$

Theorem 2.2.3. *Let $1 \leq \ell < k$. Let $\mathcal{H}$ be the (infinite) family of all ℓ-uneven k-edge hypergraphs. Then, $f(m, \mathcal{H}) \to \infty$ as $m \to \infty$. Conversely, if $\mathcal{H}$ is the family of all ℓ-even k-edge hypergraphs, we have $f(m, \mathcal{H}) = k - 1$.*

2.2.2 Non-constant Sequences

As a first step towards understanding the general problem in (2.1), we focus on the case when for every $m \geq 1$, $\mathcal{H}_m = \{H_m\}$ for a single hypergraph H_m, and further assume that all these hypergraphs H_m have the same number of edges. Thus we ask the following question:

$$\begin{gathered}\text{For which sequence of } k\text{-edge hypergraphs } \{H_m\}_{m=1}^{\infty} \\ \text{is } f(m, H_m) \text{ bounded (as } m \to \infty)?\end{gathered} \tag{2.2}$$

We are unable to answer question (2.2) completely, even for $k = 3$. Our main results provide several necessary, or sufficient conditions that partially answer (2.2). Before presenting them, we reintroduce the following crucial definition:

Definition 2.2.4 (Equal Intersection Property)**.** For $k \geq 2$, Let $\mathbf{EIP}_k$ denote the set of all k-edge hypergraphs $H = \{A_1, \ldots, A_k\}$ such that for every $1 \leq \ell \leq k$ and $I, J \in \binom{[k]}{\ell}$, we have $\left|\bigcap_{i \in I} A_i\right| = \left|\bigcap_{j \in J} A_j\right|$.

Since every two edges of a hypergraph form a 2-sunflower, we observe that the case

$k = 2$ follows immediately from the construction in Theorem 2.2.1.

Proposition 2.2.5. *Let H_m be a 2-edge hypergraph for each $m \geq 1$. Then $f(m, H_m)$ is bounded as $m \to \infty$ if and only if $H_m \in \mathbf{EIP}_2$ for all but finitely many m.*

We may therefore assume in what follows that $k \geq 3$.

Let us now fix a hypergraph $H = \{A_1, \ldots, A_k\}$ in $\mathbf{EIP}_k$. H can be encoded by k parameters $(b_1, \ldots, b_k)$, corresponding to the k distinct sizes appearing in the Venn diagram of H. More precisely, for $1 \leq \ell \leq k$, and for all $I \in \binom{[k]}{\ell}$, let

$$b_\ell := \left| \bigcap_{i \in I} A_i \setminus \bigcup_{i \in [k] \setminus I} A_i \right|.$$

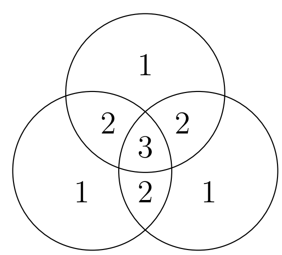

Figure 2-1: An example: $H(1, 2, 3) \in \mathbf{EIP}_3$

By inclusion-exclusion, $b_1, \ldots, b_k$ are well-defined for hypergraphs in $\mathbf{EIP}_k$. We denote $H \in \mathbf{EIP}_k$ with parameters $b_1, \ldots, b_k \geq 0$ by $H(\vec{b})$, where $\vec{b} = (b_1, \ldots, b_k)$. We shall see later (Lemma 2.4.1) that every sequence of k-edge hypergraphs $\{H_m\}$ such that $f(m, H_m)$ is bounded can only have finitely many members not in $\mathbf{EIP}_k$. For sequences $\{H_m\}_{m=1}^{\infty}$ such that $H_m \in \mathbf{EIP}_k$ for every $m \geq 1$, we obtain a sequence of length k vectors $\{\vec{b}(m)\}_{m=1}^{\infty}$, where $\vec{b}(m) = (b_1(m), \ldots, b_k(m))$. We use boldface and write $\vec{\mathbf{b}}$ for the sequence $\{\vec{b}(m)\}_{m=1}^{\infty}$.

Definition 2.2.6 ($\alpha(\vec{\mathbf{b}})$)**.** For every sequence of length k vectors $\vec{\mathbf{b}} = \{\vec{b}(m)\}_{m=1}^{\infty}$ and $m \geq 1$, let

$$\alpha(\vec{\mathbf{b}})(m) := \min_{1 \leq i \leq k-2} \left(\frac{b_i(m)}{m b_{i+1}(m)} \right).$$

Now we state our main results. To simplify notation we will often write b_i instead of $b_i(m)$ and $\alpha(\vec{\mathbf{b}})$ instead of $\alpha(\vec{\mathbf{b}})(m)$.

Theorem 2.2.7. *Let $k \geq 3$. Suppose the sequence of length k vectors $\vec{\mathbf{b}}$ satisfies $b_1, \ldots, b_{k-2} > 0$, $b_{k-1}, b_k \geq 0$ for every m. Then, for $m \geq 6$,*

$$\left(\frac{1}{2\left(\alpha(\vec{\mathbf{b}}) + \frac{1}{m}\right)\binom{b_{k-1}+b_k}{b_k}}\right)^{\frac{1}{k}} \leq f(m, H(\vec{\mathbf{b}})) \leq \frac{k(k-1)}{\alpha(\vec{\mathbf{b}})} + k - 1.$$

Theorem 2.2.7 implies that when $\binom{b_{k-1}+b_k}{b_k}$ is bounded from above, $f(m, H(\vec{\mathbf{b}}))$ is bounded from above if and only if the sequence $\alpha(\vec{\mathbf{b}})$ is bounded away from zero.

We also have the following additional lower bound on $f(m, H(\vec{\mathbf{b}}))$:

Theorem 2.2.8. *Fix $k \geq 3$. Let $\vec{\mathbf{b}} = \{\vec{b}(m)\}_{m=1}^{\infty}$ be such that $b_k(m) = b_k$ for every m. Then, for $m \geq 6$,*

$$f(m, H(\vec{\mathbf{b}})) \geq \begin{cases} m^{\frac{1}{k(b_k+1)}}\left(\frac{b_{k-1}}{4(b_{k-2}+2b_{k-1})}\right)^{\frac{1}{k}}, & k \geq 4, \\ m^{\frac{1}{b_3+2}}\left(\frac{b_2}{4(b_1+2b_2)}\right)^{\frac{b_3+1}{b_3+2}}, & k = 3. \end{cases}$$

We now focus on $k = 3$. In this case $\alpha(\vec{\mathbf{b}}) = b_1/mb_2$ and Theorem 2.2.7 reduces to

$$\left(\frac{1}{2\left(\frac{b_1}{mb_2} + \frac{1}{m}\right)\binom{b_2+b_3}{b_3}}\right)^{\frac{1}{3}} \leq f(m, H(\vec{\mathbf{b}})) \leq \frac{6mb_2}{b_1} + 2. \tag{2.3}$$

When $b_3 = 0$, (2.3) implies that $f(m, H_3(b_1, b_2, 0))$ is bounded if and only if $b_1 = \Omega(mb_2)$. We now turn to $b_3 = 1$ which already seems to be a very interesting special case that is related to an open question in extremal graph theory (see Problem 2.7.3 in Section 2.7). Here (2.3) and Theorem 2.2.8 yield the following.

Corollary 2.2.9. Let $m \to \infty$. Then $f(m, H_3(b_1, b_2, 1))$ is bounded when $b_1 = \Omega(mb_2)$ and it is unbounded when either $b_1 + b_2 = o(m)$ or $b_1 = o(\sqrt{m}\, b_2)$.

Corollary 2.2.9 can be summarized in Figure 2-2. The light region corresponds to a bounded $f(m, H(\vec{\mathbf{b}}))$, and the dark region corresponds to unbounded $f(m, H(\vec{\mathbf{b}}))$. White regions correspond to areas where we do not know if $f(m, H(\vec{\mathbf{b}}))$ is bounded or not.

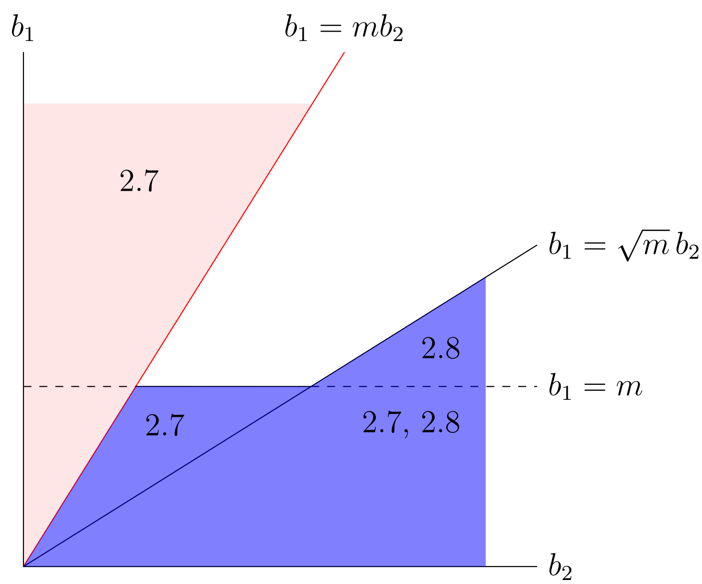

Figure 2-2: Theorems 2.2.7 and 2.2.8 for $\vec{\mathbf{b}} = (b_1, b_2, 1)$

We are able to refine our results slightly via the following result.

Theorem 2.2.10. *For every odd prime power q we have*

$$f(q^2 + 1, H(q^2 - q - 1, q, 1)) = 2.$$

For functions $f(m)$ and $g(m)$, we write $f \gg g$ iff $g = o(f)$. Later, we shall show that Theorem 2.2.10 implies the following.

Corollary 2.2.11. When $b_1 \geq b_2^2$, $b_2 \geq \sqrt{m}$ and b_2 is a prime power,

$$f(m, H_3(b_1, b_2, 1)) = 2. \tag{2.4}$$

Further, when $b_1 \gg b_2^2$ and $b_2 \geq m^{0.68}$,

$$f(m, H_3(b_1, b_2, 1)) = 2. \tag{2.5}$$

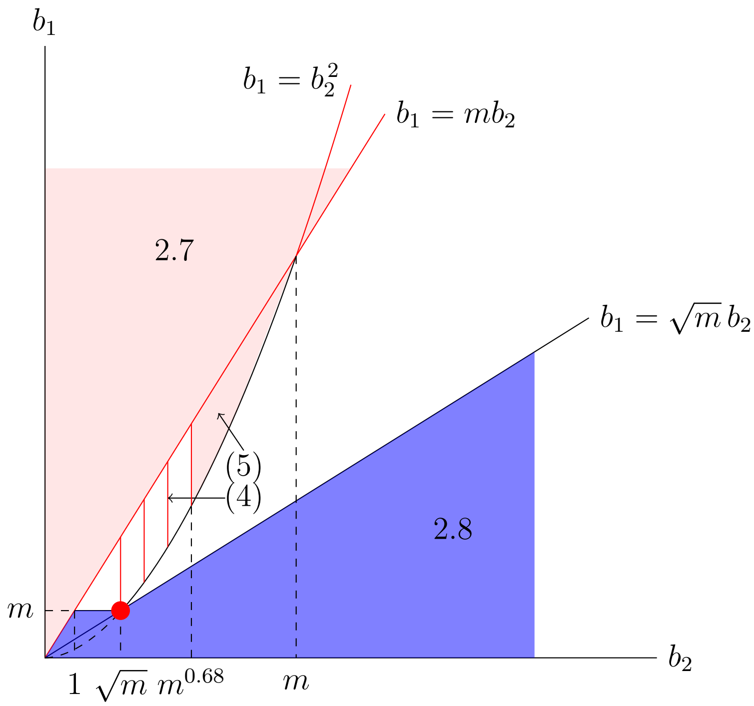

Figure 2-3: $\vec{\mathbf{b}} = (b_1, b_2, 1)$

The improvement that Corollary 2.2.11 yields on Figure 2-2 is summarized in Figure 2-3. Note that we are using the parabola $b_1 = b_2^2$ as an asymptotic approximation of Corollary 2.2.11. By (2.4), $f(m, H_3(b_1, b_2, 1)) = 2$ infinitely often on this parabola, figuratively represented by vertical stripes in the interval $\sqrt{m} \leq b_2 \leq m^{0.68}$. We shall see later, by virtue of Theorem 2.7.2, that in the white region to the right of $b_1 = b_2^2$ and between the lines $b_1 = mb_2$ and $b_1 = \sqrt{m}\, b_2$, we have $f(m, H_3(b_1, b_2, 1)) > 2$.

2.3 Proofs of Theorems 2.2.1 and 2.2.3

In this section, we prove Theorems 2.2.1 and 2.2.3, which answer question (2.1) for constant sequences. We use the following well-known facts about sunflowers and diagonal hypergraph Ramsey numbers.

Recall that a q-sunflower is a hypergraph $\{A_1, \ldots, A_q\}$ such that $A_i \cap A_j = \bigcap_{s=1}^{q} A_s$. The celebrated Erdős-Rado sunflower Lemma [25] states the following.

Lemma 2.3.1 (Erdős-Rado)**.** *Let H be an r-graph with $|H| = r!(\alpha - 1)^r$. Then, H contains an α-sunflower.*

Next, recall that the hypergraph Ramsey number $r_\ell(s,t)$ is the minimum N such that any ℓ-graph on N vertices, admits a clique of size s or an independent set of size t. The following is a well-known theorem of Erdős, Hajnal and Rado [23]:

Theorem 2.3.2. *There are absolute constants $c(\ell), c'(\ell)$ such that*

$$twr_{\ell-1}(c't^2) < r_\ell(t,t) < twr_\ell(ct).$$

Here the tower function $twr_k(x)$ is defined by $twr_0(x) = 1$ and $twr_{i+1}(x) = 2^{twr_i(x)}$.

The right side of this theorem can be rewritten as follows:

> Let F be any ℓ-graph on n vertices. Then there is an absolute constant c_ℓ such that there is a subgraph $F' \subset F$ with $|V(F')| \geq c_\ell \cdot \log_{(\ell)}(n)$, which is either a clique or an independent set. Here $\log_{(\ell)}$ denotes iterated logarithms. (2.6)

Now we are prepared to prove Theorems 2.2.1 and 2.2.3. Recall that a hypergraph is uniform if all its edges have the same size, otherwise it is non-uniform.

Proof of Theorem 2.2.1. Fix a family of hypergraphs $\mathcal{H}$ with n members, say $\mathcal{H} = \{H_1, \ldots, H_n\}$. Let $H_i \in \mathcal{H}$ be an r-uniform q-sunflower with core W. For every

$m \geq q$, let F be an r-uniform m-sunflower with core W. Then every subset of F of size q is isomorphic to H_i, thus proving $f(m, \mathcal{H}) \leq q - 1$.

Suppose now that $\mathcal{H}$ consists of ℓ many uniform hypergraphs labeled $H_1, \ldots, H_\ell$ (none of which are sunflowers), and $(n - \ell)$ many non-uniform hypergraphs labeled $H_{\ell+1}, \ldots, H_n$. For $1 \leq i \leq \ell$, let r_i be the uniformity of H_i. Given any hypergraph F with m edges, we find a large $\mathcal{H}$-free subgraph as follows. First, since H_n is non-uniform, it contains a set of size a and a set of size $b \neq a$. Clearly, at least half of the edges of F have size $\neq a$, or at least half of them have size $\neq b$. Take the appropriate subgraph $F_1 \subset F$ of size $\geq \frac{m}{2}$. By successively halving the sizes, we obtain a chain of hypergraphs $F_{n-\ell} \subset F_{n-\ell-1} \subset \cdots \subset F_1 \subset F$ such that $F_{n-\ell}$ is $\{H_{\ell+1}, \ldots, H_n\}$-free, and $|F_{n-\ell}| \geq \frac{m}{2^{n-\ell}}$.

We now deal with the uniform part of $\mathcal{H}$. Notice that by Lemma 2.3.1, any r-graph G with $|G| = m$ contains an α-sunflower, as long as $m > r!\alpha^r$. Taking $\alpha = \lfloor c_r m^{1/r} \rfloor$ where $c_r = ((2r)!)^{-1/r}$, satisfies the required condition. So, every r-graph G of size m contains a sunflower of size $\lfloor c_r m^{1/r} \rfloor$.

Since H_ℓ is r_ℓ-uniform, we note that either $F_{n-\ell}$ contains a subgraph of size $\frac{1}{2}|F_{n-\ell}|$ which has no sets of size r_ℓ (and hence is H_ℓ-free), or there is a subgraph of size $\frac{1}{2}|F_{n-\ell}|$ which is r_ℓ-uniform. In the second case, using Lemma 2.3.1 on this subgraph, we obtain an H_ℓ-free subgraph of $F_{n-\ell}$ of size at least $c_{r_\ell}\left(\frac{m}{2^{n-\ell+1}}\right)^{\frac{1}{r_\ell}}$. Thus, in either case, we conclude that there exists an H_ℓ-free subgraph $F'_{n-\ell+1} \subset F_{n-\ell}$ such that

$$|F'_{n-\ell+1}| \geq \min\left\{\frac{m}{2^{n-\ell+1}} \ , \ c_{r_\ell}\left(\frac{m}{2^{n-\ell+1}}\right)^{\frac{1}{r_\ell}}\right\} \geq c'_{\mathcal{H}} \cdot m^{\frac{1}{r_\ell}}.$$

We iterate the same argument $\ell - 1$ more times, to finally obtain a constant $C_{\mathcal{H}}$ and a subgraph $F'_\ell \subset F_{n-\ell}$ such that F'_ℓ is $\mathcal{H}$-free, and

$$|F'_\ell| \geq C_{\mathcal{H}} \cdot m^{\frac{1}{r_1 \cdots r_\ell}}.$$

□

Proof of Theorem 2.2.3. Let $F = \{F_1, \ldots, F_m\}$ have size m. Suppose $1 \le \ell < k$, and $\mathcal{H}$ is the family of all ℓ-uneven k-graphs. Then, there are distinct subsets $I, J \in \binom{[k]}{\ell}$, such that for every $H = \{A_1, \ldots, A_k\} \in \mathcal{H}$, $\bigcap_{i \in I} A_i = \varnothing$ and $\bigcap_{j \in J} A_j \ne \varnothing$. Then, we construct an ℓ-graph G with vertex set F, and hyperedges $\{\{F_1, \ldots, F_\ell\} : F_1 \cap \cdots \cap F_\ell = \varnothing\}$. By (2.6), there is a a constant c_ℓ and a subset $F' \subseteq F$ of size $\ge c_\ell \cdot \log_{(\ell)}(m)$, such that F' is either a clique or an independent set in G. In either case, F' is $\mathcal{H}$-free.

On the other hand, suppose $\mathcal{H}$ is such that for some $1 \le \ell \le k$ and any $I \subseteq [k]$, $\bigcap_{i \in I} A_i \ne \varnothing$ iff $|I| \le \ell$. For every $m \ge k$, we construct a hypergraph $F = \{F_1, \ldots, F_m\}$ in the following manner. Consider the bipartite graph $B = \left([m], \binom{[m]}{\ell}\right)$ where $x \in [m]$ is adjacent to $y \in \binom{[m]}{\ell}$ iff $x \in y$. Let F_i be the set of neighbors in B of the vertex $i \in [m]$. Notice that for any $I \subseteq [k]$,

$$\bigcap_{i \in I} F_i = \begin{cases} \varnothing, & |I| > \ell, \\ \ne \varnothing, & |I| \le \ell. \end{cases}$$

This construction therefore shows that $f(m, \mathcal{H}) = k - 1$. □

2.4 Proof of Theorem 2.2.7

In this section, we prove Theorem 2.2.7. We begin with some preliminary analysis of the family $\mathbf{EIP}_k$.

First, we make the crucial observation regarding question (2.2) that every sequence of k-edge hypergraphs $\{H_m\}$ such that $f(m, H_m)$ is bounded, can only have finitely many members not in $\mathbf{EIP}_k$. This follows immediately from Lemma 2.4.1. Furthermore, for any $H(\vec{b}) \in \mathbf{EIP}_k$, one can explicitly determine the relation between the intersection sizes and the parameters $b_1, \ldots, b_k$ by inclusion-exclusion. We state this relation in Lemma 2.4.2.

Lemma 2.4.1. *Suppose $H = \{A_1, \ldots, A_k\}$ satisfies the following for some $1 \le \ell \le k$:*

there are two sets of indices $I, J \in \binom{[k]}{\ell}$ *such that* $|\bigcap_{i \in I} A_i| = a$ *and* $|\bigcap_{j \in J} A_j| = b$ *with* $a \neq b$. *Then there is a constant* c_ℓ *such that* $f(m, H) \geq c_\ell \cdot \log_{(\ell)}(m)$.

Proof of Lemma 2.4.1. Let F be any hypergraph with m edges. Construct an ℓ-graph G with F as its vertex set, and hyperedges

$$\{\{B_1, \dots, B_\ell\} : |B_1 \cap \dots \cap B_\ell| = a\}.$$

By (2.6), there exists a subset $F' \subseteq F$ of size $c_\ell \cdot \log_{(\ell)}(m)$ which is either a clique or an independent set in G. In either case, H cannot be contained in F'. ■

Lemma 2.4.1 implies that if there are infinitely many m such that $H_m \notin \mathbf{EIP}_k$, then for each such non-EIP hypergraphs we have $f(m, H_m) \geq c' \cdot \log_{(k)}(m)$, where c' is the absolute constant $c' = \min\{c_1, \dots, c_k\}$. This is an infinite subsequence of $\{H_m\}$. Therefore, if $f(m, H_m)$ is bounded, then by looking at the tail of $\{H_m\}$, we may assume WLOG that $H_m \in \mathbf{EIP}_k$ for every $m \geq 1$.

Recall that hypergraphs $H \in \mathbf{EIP}_k$ are characterized by the length k-vector $\vec{b}$, and for every sequence of hypergraphs $\{H_m\}_{m=1}^{\infty}$, we have a corresponding sequence of length k vectors $\vec{\mathbf{b}}$.

We now state the relation between the intersection sizes and the parameters $b_1, \dots, b_k$ for $H(\vec{b}) \in \mathbf{EIP}_k$.

Lemma 2.4.2. *Let* $H(\vec{b}) \in \mathbf{EIP}_k$, *and* $a_i = |A_1 \cap \dots \cap A_i|$, *for each* $1 \leq i \leq k$. *Then,*

$$b_i = a_i - \binom{k-i}{1} a_{i+1} + \binom{k-i}{2} a_{i+2} - \dots + (-1)^{k-i} \binom{k-i}{k-i} a_k. \tag{2.7}$$

Before proving Theorem 2.2.7, we prove an auxiliary upper bound in Lemma 2.4.3, which provides a better upper bound on $f(m, H(\vec{\mathbf{b}}))$ with tighter constraints on $\vec{\mathbf{b}}$.

Lemma 2.4.3. *Suppose* $\vec{\mathbf{b}} = (b_1, \dots, b_k)$ *is such that* $b_i \geq 0$, *and for every* $1 \leq i \leq$

$k-1$,

$$\sum_{j=i}^{k-1}(-1)^{j-i}\binom{m-k+j-i-1}{j-i}b_j \geq 0. \tag{2.8}$$

Then $f(m, H(b_1, \ldots, b_k)) = k-1$.

Proof of Lemma 2.4.3. Let $\vec{\mathbf{b}}$ satisfy the restrictions given in (2.8). Note that we need to construct a hypergraph sequence $\{F_m\}_{m=1}^{\infty}$, such that every k-edge subgraph of F_m is isomorphic to $H(\vec{b})$. To achieve this, we define the following general construction:

Construction 2.4.4 **($F_m^{d_1,\ldots,d_k}$).** Given $d_1, \ldots, d_k \geq 0$ and $m \geq k$, let $B = ([m], Y)$ be the bipartite graph with parts $[m]$ and Y, where Y is defined as follows. For $1 \leq \ell \leq k$ and $1 \leq j \leq d_\ell$, let

$$Y_j^\ell = \left\{ \begin{array}{cc} \{v_j^S : S \in \binom{[m]}{\ell}\}, & \ell < k \\ \{w_j\}, & \ell = k \end{array} \right\},$$

where $v_j^S \neq v_{j'}^{S'}$ for every $(j, S) \neq (j', S')$ and $w_j \neq w_{j'}$ for every $j \neq j'$. Then

$$Y = \bigcup_{\ell=1}^{k} \bigcup_{j=1}^{d_\ell} Y_j^\ell.$$

For $x \in [m]$ and $v_j^S \in Y$, let $(x, v_j^S) \in E(B)$ iff $x \in S$, and let $(x, w_j) \in E(B)$ for every $x \in [m]$ and $w_j \in Y$. Then, define $F_m^{d_1,\ldots,d_k} = \{A_1, \ldots, A_m\}$, where $A_i = N_B(i) \subset Y$ for $i = 1, \ldots, m$. ■

For example, the construction $F_4^{1,2,3}$ is given by:

$$\left\{ \begin{array}{l} A_1 = \left\{v_1^1; v_1^{12}, v_2^{12}, v_1^{13}, v_2^{13}, v_1^{14}, v_2^{14}; w_1, w_2, w_3\right\} \\ A_2 = \left\{v_1^2; v_1^{12}, v_2^{12}, v_1^{23}, v_2^{23}, v_1^{24}, v_2^{24}; w_1, w_2, w_3\right\} \\ A_3 = \left\{v_1^3; v_1^{13}, v_2^{13}, v_1^{23}, v_2^{23}, v_1^{34}, v_2^{34}; w_1, w_2, w_3\right\} \\ A_4 = \left\{v_1^4; v_1^{14}, v_2^{14}, v_1^{24}, v_2^{24}, v_1^{34}, v_2^{34}; w_1, w_2, w_3\right\} \end{array} \right\}.$$

Informally, in this example, A_i consists of one vertex v_1^i corresponding to $\{i\}$, two vertices v_1^{ij} and v_2^{ij} corresponding to two-element subsets $\{i, j\}$, and three vertices w_1, w_2, w_3 that are in the common intersection of all the A_i's, $1 \le i \le 4$.

We observe the following property of the intersection sizes of the edges of $F_m^{d_1,\dots,d_k}$.

Claim 2.4.5. For $1 \le i \le k$ and any i-edge subgraph $\{A_{r_1}, \dots, A_{r_i}\} \subset F_m^{d_1,\dots,d_k}$, the size of the common intersection $a_i := |A_{r_1} \cap \cdots \cap A_{r_i}|$ is given by

$$a_i = d_i + \binom{m-i}{1} d_{i+1} + \cdots + \binom{m-i}{k-1-i} d_{k-1} + d_k. \tag{2.9}$$

Proof of Claim 2.4.5. Suppose $G = \{A_{r_1}, \dots, A_{r_i}\} \subset F_m^{d_1,\dots,d_k}$. We shall now count $|A_{r_1} \cap \cdots \cap A_{r_i}|$. For a fixed hypergraph $F_m^{d_1,\dots,d_k} \supseteq G' \supseteq G$, let $U_{G'}$ denote the set of all vertices of $F_m^{d_1,\dots,d_k}$ which are in all the edges of G' but none of the edges of $F_m^{d_1,\dots,d_k} \setminus G'$. Notice that $A_{r_1} \cap \cdots \cap A_{r_i}$ is a disjoint union of $U_{G'}$'s, $G' \supseteq G$. Therefore,

$$a_i = |A_{r_1} \cap \cdots \cap A_{r_i}| = \sum_{G' \supseteq G} |U_{G'}| = \sum_{G' \supseteq G} \left| \bigcap_{X \in G'} X \setminus \bigcup_{X \notin G'} X \right|. \tag{2.10}$$

Fix a $G' \supseteq G$. Let $G' = \{A_{r_1}, \dots, A_{r_i}, A_{s_1}, \dots, A_{s_{|G'|-i}}\}$. We observe that,

- For $i \le |G'| < k$, $U_{G'}$ consists exactly of the vertices

$$\left\{ v_j^{\{r_1,\dots,r_i,s_1,\dots,s_{|G'|-1}\}} : 1 \le j \le d_{|G'|} \right\}.$$

- For $k \le |G'| < m$, $\bigcap_{X \in G'} X = \{w_1, \dots, w_{d_k}\} \subseteq \bigcup_{X \notin G'} X$, thus

$$U_{G'} = \varnothing.$$

- For $|G'| = m$, $U_{G'} = \bigcap_{X \in G'} X = \{w_1, \dots, w_{d_k}\}$.

Therefore,

$$|U_{G'}| = \begin{cases} d_{|G'|}, & i \le |G'| < k, \\ 0, & k \le |G'| < m, \\ d_k, & |G'| = m. \end{cases}$$

Plugging back these values into (2.10), we get

$$a_i = d_i + \binom{m-i}{1} d_{i+1} + \cdots + \binom{m-i}{k-1-i} d_{k-1} + d_k$$

for every $1 \leq i \leq k$. ■

Now we return to the proof of Lemma 2.4.3. Given a length k vector $\vec{b} \geq 0$ which satisfies (2.8) for $1 \leq i \leq k-1$, let d_i be the left hand side of (2.8), i.e.,

$$d_i := \sum_{j=i}^{k-1} (-1)^{j-i} \binom{m-k+j-i-1}{j-i} b_j,$$

and let $d_k = b_k$. Now, we look at the construction $F_m = F_m^{d_1,\ldots,d_k}$, and pick any k-edge subgraph $G \subset F_m$. Observe that $G \in \mathbf{EIP}_k$, and therefore there is a length k vector $\vec{g}$ such that $G = H(\vec{g})$. It suffices to check that $\vec{g} = \vec{b}$.

Suppose $G = \{A_1, \ldots, A_k\}$. For $1 \leq i \leq k$, let $a_i := |A_1 \cap \cdots \cap A_i|$. Recall that Lemma 2.4.2 gave us a way of computing $\vec{g}$ in terms of $\vec{a}$, and Claim 2.4.5 computes $\vec{a}$ in terms of $\vec{d}$. In order to precisely write down these relations, we introduce a few matrices.

Notation. Let us define the following quantities for arbitrary $m \geq k \geq 1$.

- Let $a_{ij}^{(m)} = \binom{m-i}{j-i}$ and $b_{ij}^{(m)} = (-1)^{j-i}\binom{m-i}{j-i}$.* Then, denote by $A_{k,m}$ and $B_{k,m}$ the upper triangular matrices

$$A_{k,m} = (a_{ij}^{(m)})_{1 \leq i,j \leq k}, \text{ and } B_{k,m} = (b_{ij}^{(m)})_{1 \leq i,j \leq k},$$

- Let $\vec{\mathbf{1}}$ denote the all-one vector, and $\vec{\mathbf{0}}$ the all-zero vector.
- Define $D_{k-1,m} := \begin{bmatrix} A_{k-1,m} & \vec{\mathbf{1}} \\ \vec{\mathbf{0}}^\intercal & 1 \end{bmatrix}$.

*By our convention, $\binom{x}{y} = 0$ if $y < 0$. Thus $a_{ij}^{(m)} = b_{ij}^{(m)} = 0$ whenever $j < i$.

- Let $W_{k-1,m}$ be the $(k-1)\times(k-1)$ matrix given by

$$W_{k-1,m} = (w_{ij}^{(m)})_{1\le i,j\le k-1},$$

where $w_{ij}^{(m)} = (-1)^{j-i}\binom{m-k+j-i-1}{j-i}$.

- Define $W'_{k-1,m} := \begin{bmatrix} W_{k-1,m} & \vec{\mathbf{0}} \\ \vec{\mathbf{0}}^{\mathsf{T}} & 1 \end{bmatrix}$. ■

First, we observe that the assertion of Lemma 2.4.2 can be rephrased as,

$$\vec{g} = B_{k,k}\vec{a}. \tag{2.11}$$

Next, in terms of matrices, equality (2.9) reads

$$\vec{a} = D_{k-1,m}\vec{d}. \tag{2.12}$$

Finally, by the definition of $\vec{d}$, we have

$$\vec{d} = W'_{k-1,m}\vec{b}. \tag{2.13}$$

Putting together Equations (2.11,2.12,2.13), we obtain:

$$\vec{g} = B_{k,k}D_{k-1,m}W'_{k-1,m}\cdot\vec{b}.$$

By Proposition A.2 from Appendix A, we know that the matrix $B_{k,k}D_{k-1,m}W'_{k-1,m}$ is I_k, and this concludes the proof of Lemma 2.4.3. ■

We now have gathered all the equipment required to complete the proof of Theorem 2.2.7.

Proof of Theorem 2.2.7. Recall that $\alpha = \min_{1\le i\le k-2}\left(\frac{b_i(m)}{mb_{i+1}(m)}\right)$, and we wish to prove that

$$f(m, H(\vec{\mathbf{b}})) \leq \frac{k(k-1)}{\alpha} + k - 1.$$

Note that this bound is trivial if $\frac{k(k-1)}{\alpha} \geq m$, therefore we may assume that $\alpha m > k(k-1)$. From the definition of α, note that $b_i \geq \alpha m b_{i+1}$ for each $1 \leq i \leq k-2$. By successively applying these inequalities we obtain $b_i \geq \alpha m b_{i+1} \geq \alpha^2 m^2 b_{i+2} \geq \cdots \geq \alpha^{k-i-1} m^{k-i-1} b_{k-1}$. Thus,

$$\begin{aligned} b_i \geq \alpha m b_{i+1} &\geq \sum_{r=i+1}^{k-1} \frac{\alpha m}{k} \cdot b_{i+1} \\ &\geq \sum_{r=i+1}^{k-1} \frac{\alpha^{r-i} m^{r-i}}{k} \cdot b_r \\ &\geq \sum_{r=i+1}^{k-1} \left(\frac{\alpha m}{k}\right)^{r-i} b_r \\ &\geq \sum_{r=i+1}^{k-1} \binom{\lfloor \frac{\alpha m}{k} \rfloor}{r-i} b_r. \end{aligned} \tag{2.14}$$

The last inequality follows from $X^t \geq \binom{\lfloor X \rfloor}{t}$. Observe that the assumption $\frac{\alpha m}{k} > k-1$ implies $\lceil \frac{\alpha m}{k} \rceil \geq k$. Therefore, for $1 \leq i \leq k-2$ and $i+1 \leq r \leq k-1$, we have

$$\left\lfloor \frac{\alpha m}{k} \right\rfloor \geq \left\lceil \frac{\alpha m}{k} \right\rceil - k + r - i - 1 \geq 0.$$

Thus, (2.14) gives us

$$\begin{aligned} b_i \geq \sum_{r=i+1}^{k-1} \binom{\lfloor \frac{\alpha m}{k} \rfloor}{r-i} b_r &\geq \sum_{r=i+1}^{k-1} \binom{\lceil \frac{\alpha m}{k} \rceil - k + r - i - 1}{r-i} b_r \\ &\geq \sum_{r=i+1}^{k-1} (-1)^{r-i+1} \binom{\lceil \frac{\alpha m}{k} \rceil - k + r - i - 1}{r-i} b_r, \end{aligned}$$

implying

$$b_i + \sum_{r=i+1}^{k-1} (-1)^{r-i} \binom{\lceil \frac{\alpha m}{k} \rceil - k + r - i - 1}{r-i} b_r \geq 0.$$

This is exactly the condition (2.8), with m replaced by $\lceil \frac{\alpha m}{k} \rceil$, so Lemma 2.4.3 gives us a hypergraph K on $\lceil \frac{\alpha m}{k} \rceil$ edges such that every k sets of K are isomorphic to $H(\vec{\mathbf{b}})$.

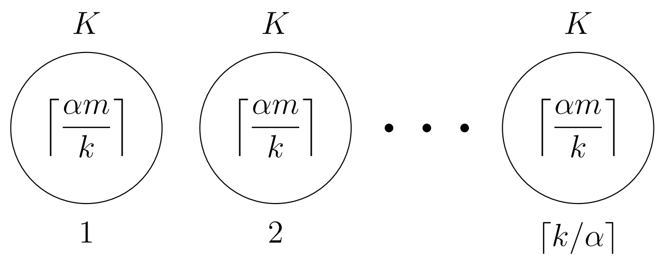

Figure 2-4: Constructing F_m from copies of K

Now, consider a $\lceil \frac{k}{\alpha} \rceil$-fold disjoint union of K's. This hypergraph F_m has $\lceil \frac{k}{\alpha} \rceil \cdot \lceil \frac{\alpha m}{k} \rceil \geq m$ edges, and note that as long as we pick $1 + \lceil \frac{k}{\alpha} \rceil \cdot (k-1)$ edges, some k of them fall in the same copy of K. These k edges create a $H(\vec{\mathbf{b}})$ by construction of K. This shows $f(m, H(\vec{\mathbf{b}})) \leq \lceil \frac{k}{\alpha} \rceil \cdot (k-1)$, completing the proof of the upper bound.

Now we prove the lower bound. Recall that we are aiming to prove

$$f(m, H(\vec{\mathbf{b}})) \geq \max_{1 \leq i \leq k-2} \left(\frac{m b_{i+1}}{2(b_i + b_{i+1}) \binom{b_{k-1}+b_k}{b_k}} \right)^{\frac{1}{k}}. \tag{2.15}$$

Suppose F is a hypergraph on m edges. Either F has a subgraph F_1 of size $\frac{m}{2}$ which is of the same uniformity as $H(\vec{b})$, or it has a subgraph of size $\frac{m}{2}$ which is not of this uniformity. If the latter is true, then $\mathrm{ex}(F, H(\vec{b})) \geq \frac{m}{2}$. Otherwise, we focus on the subgraph F_1. Let T be a $H(\vec{b})$-free subgraph in F_1 of maximum size, say $|T| = t$. Then, for every $S \in F_1 \setminus T$, there exist distinct $A_1, \ldots, A_{k-1} \in T$ such that $\{A_1, \ldots, A_{k-1}, S\}$ forms a $H(\vec{b})$. Therefore, there are fixed $A_1, \ldots, A_{k-1} \in T$ and a subgraph $F_2 \subseteq F_1 \setminus T$ such that $\{A_1, \ldots, A_{k-1}, S\}$ forms a $H(\vec{b})$ for every $S \in F_2$, where

$$|F_2| \geq \frac{\frac{m}{2} - t}{\binom{t}{k-1}}.$$

Further, note that $|A_1 \cap \cdots \cap A_{k-1} \cap S| = b_k$ for every $S \in F_2$, therefore there is a subgraph $F_3 \subseteq F_2$ such that every element $S \in F_3$ intersects $A_1 \cap \cdots \cap A_{k-1}$ in the exact same set, and

$$|F_3| \geq \frac{\frac{m}{2} - t}{\binom{t}{k-1}\binom{b_{k-1}+b_k}{b_k}}.$$

Finally, for any $1 \leq i \leq k-2$, let $X_i := A_1 \cap \cdots \cap A_i \setminus (A_{i+1} \cup \cdots \cup A_{k-1})$, and

$$h_i := |\{(x, B) : \ x \in X_i, \ B \in F_3, \ x \in B\}|.$$

Let $D := \max_{x \in V(F_3)} \deg_{F_3}(x)$. As $\{A_1, \ldots, A_{k-1}, B\}$ is an $H(\vec{b})$ for each $B \in F_3$,

$$|F_3| \cdot b_{i+1} = h_i \leq D \cdot |X_i|. \tag{2.16}$$

Now, for a fixed $S \in F_3$,

$$\begin{aligned} |X_i| &= |S \cap X_i| + |X_i \setminus S| \\ &= \left| S \cap \bigcap_{j=1}^{i} A_j \setminus \left(\bigcup_{j=i+1}^{k-1} A_j \right) \right| + \left| \bigcap_{j=1}^{i} A_j \setminus \left(\bigcup_{j=i+1}^{k-1} A_j \cup S \right) \right| \\ &= b_{i+1} + b_i, \end{aligned}$$

Therefore (2.16) implies

$$D \geq \frac{|F_3| \cdot b_{i+1}}{b_i + b_{i+1}} \geq \frac{(\frac{m}{2} - t) b_{i+1}}{\binom{t}{k-1}\binom{b_{k-1}+b_k}{b_k}(b_i + b_{i+1})}.$$

Note that the sets in F_3 that achieve the maximum degree D is $H(\vec{b})$-free. This is because if I is the common intersection of any set from F_3 with $A_1 \cap \cdots \cap A_{k-1}$, and if x is a vertex of degree D in F_3, then every edge through x contains $\{x\} \cup I$. This leads us to the inequality

$$t \geq \frac{(\frac{m}{2} - t) b_{i+1}}{\binom{t}{k-1}(b_i + b_{i+1})\binom{b_{k-1}+b_k}{b_k}},$$

i.e.,

$$t\binom{t}{k-1} \geq \frac{(\frac{m}{2} - t) b_{i+1}}{(b_i + b_{i+1})\binom{b_{k-1}+b_k}{b_k}}.$$

Since $m \geq 6$, note that if $t \geq \frac{m}{4}$, then $t \geq \left(\frac{m}{2}\right)^{\frac{1}{3}} \geq \left(\frac{m}{2}\right)^{\frac{1}{k}}$, which is larger than the right side of (2.15). So we may assume $t < \frac{m}{4}$, which would lead us to

$$t^k \geq 2t\binom{t}{k-1} \geq \frac{mb_{i+1}}{2(b_i + b_{i+1})\binom{b_{k-1}+b_k}{b_k}}. \tag{2.17}$$

As (2.17) holds for every $1 \leq i \leq k-2$, this gives the bound that we seek. □

2.5 Proof of Theorem 2.2.8

In this section we prove Theorem 2.2.8. The proof is by induction on b_k, starting from $b_k = 0$. Notice that the lower bound of Theorem 2.2.7 gives us the following corollary, which serves as the base case for our induction argument:

Corollary 2.5.1. For $m \geq 6$,

$$f(m, H(b_1, \ldots, b_{k-1}, 0)) \geq \max_{1 \leq i \leq k-2} \left(\frac{mb_{i+1}}{2(b_i + b_{i+1})}\right)^{\frac{1}{k}}.$$

Further, one can asymptotically improve this bound when $k = 3$:

Proposition 2.5.2. *For $m \geq 4$,*

$$f(m, H(b_1, b_2, 0)) \geq \sqrt{\frac{mb_2}{2(b_1 + 2b_2)}}.$$

Proof. Let $|F| = m$ and $H = H(b_1, b_2, 0)$. Either F has a $(b_1 + 2b_2)$-uniform subgraph F_1 of size $\frac{m}{2}$, or it has a subgraph of size $\frac{m}{2}$ in which none of the edges have size $(b_1 + 2b_2)$. If the latter is true, then $\text{ex}(F, H) \geq \frac{m}{2}$. Otherwise let us focus on F_1. Let T be an H-free subset of maximum size in F_1, and suppose $|T| = t$. Note that for any $B \in F_1 \setminus T$, there are sets $A_1, A_2 \in T$ such that (B, A_1, A_2) is a $H(b_1, b_2, 0)$. Suppose $V = \bigcup_{A \in T} A$, then we have $|B \cap V| \geq 2b_2$, and $|V| \leq t(b_1 + 2b_2)$. Let $D = \max_{x \in V} \deg_{F_1}(x)$. Then,

$$2b_2 \cdot |F_1 \setminus T| \leq |\{(x, B) : \ x \in V, B \in F_1 \setminus T, x \in B\}| \leq D \cdot |V|,$$

and

$$D \geq \frac{(m-2t)b_2}{t(b_1+2b_2)}.$$

Let $x \in V$ have the maximum degree in F. Since the subgraph of size D containing x is H-free, we obtain

$$t \geq \frac{(m-2t)b_2}{t(b_1+2b_2)}.$$

If $t \geq \frac{m}{4}$, then $t \geq \frac{1}{2}\sqrt{m} \geq \sqrt{\frac{mb_2}{2(b_1+2b_2)}}$. So assume $t < \frac{m}{4}$, and therefore $t^2 \geq \frac{mb_2}{2(b_1+2b_2)}$, as desired. □

Before we prove Theorem 2.2.8 we require the following lemma from [67]:

Lemma 2.5.3. *Let $H = (V, E)$ be a k-graph on m vertices, and let $\alpha(H)$ denote the independence number of H. Then,*

$$\alpha(H) \geq \frac{k-1}{k} \cdot \left(\frac{m^k}{k|E(H)|}\right)^{\frac{1}{k-1}}.$$

Now we are prepared to prove Theorem 2.2.8.

Proof of Theorem 2.2.8. Fix k and $\vec{\mathbf{b}}$. Recall that b_k is fixed, and we wish to show that for $m \geq 6$,

$$f(m, H(b_1,\ldots,b_k)) \geq \begin{cases} m^{\frac{1}{k(b_k+1)}}\left(\frac{b_{k-1}}{4(b_{k-2}+2b_{k-1})}\right)^{\frac{1}{k}}, & k \geq 4, \\ m^{\frac{1}{b_3+2}}\left(\frac{b_2}{4(b_1+2b_2)}\right)^{\frac{b_3+1}{b_3+2}}, & k = 3. \end{cases} \tag{2.18}$$

Suppose $|F| = m$. Then, either F has a subgraph F_1 of size at least $\frac{m}{2}$ which has uniformity the same as that of $H(\vec{b})$, or it does not. When the latter is true, we have $\mathrm{ex}(F, H(\vec{b})) \geq \frac{m}{2}$. Since $\frac{m}{2} \geq m^{\frac{1}{4}} \cdot \left(\frac{1}{8}\right)^{\frac{1}{4}}$ and $\frac{m}{2} \geq m^{\frac{1}{2}} \cdot \left(\frac{1}{8}\right)^{\frac{1}{2}}$, we may assume that the former is true. We wish to show that F_1 contains a $H(\vec{b})$-free subgraph of large size.

We proceed by induction on b_k. Notice that we already established the results for $b_k = 0$ in Corollary 2.5.1 (using $b_{k-1} \leq 2b_{k-1}$) and Proposition 2.5.2.

Construct a k-graph G with vertex set F_1 and call $\{A_1, \ldots, A_k\}$ an edge in G iff $\{A_1, \ldots, A_k\} \cong H(\vec{b})$. Clearly, $t = \alpha(G)$ is a lower bound to our problem. By Lemma 2.5.3,

$$k|E(G)| \geq \left(\frac{k-1}{k}\right)^{k-1} \cdot \frac{(m/2)^k}{t^{k-1}}.$$

Given $1 \leq i \leq k$ and $B_1, \ldots, B_i \in F_1$, denote by $\deg_G(B_1, \ldots, B_i)$ the number of edges of G containing $\{B_1, \ldots, B_i\}$. As

$$\sum_{A_1,\ldots,A_{k-2} \in F_1} \deg_G(A_1, \ldots, A_{k-2}) = \binom{k}{2}|E(G)|,$$

we obtain

$$\begin{aligned}\sum_{A_1,\ldots,A_{k-2} \in F_1} \deg_G(A_1, \ldots, A_{k-2}) &\geq \frac{\binom{k}{2}}{k} \cdot \frac{(k-1)^{k-1}}{k^{k-1}} \cdot \frac{(m/2)^k}{t^{k-1}} \\ &= \frac{(k-1)^k}{2k^{k-1}} \cdot \frac{(m/2)^k}{t^{k-1}}.\end{aligned}$$

The sum on the left side has at most $\binom{m/2}{k-2} \leq \frac{(m/2)^{k-2}}{(k-2)!}$ terms, therefore there exist distinct $A_1, \ldots, A_{k-2} \in F_1$ such that

$$\deg_G(A_1, \ldots, A_{k-2}) \geq \frac{(k-2)!(k-1)^k}{2k^{k-1}} \cdot \frac{(m/2)^2}{t^{k-1}}.$$

Note that $\frac{(k-2)!(k-1)^k}{2k^{k-1}} > \frac{1}{4}$ for every $k \geq 3$. Let $\mathcal{B}$ denote the set of all edges $B \in F_1$ which are covered by an edge through $\{A_1, \ldots, A_{k-2}\}$ in G. Then, $|\mathcal{B}|^2 \geq \deg_G(A_1, \ldots, A_{k-2})$, and so

$$|\mathcal{B}|^2 \geq \frac{1}{4} \cdot \frac{(m/2)^2}{t^{k-1}} = \frac{1}{16} \cdot \frac{m^2}{t^{k-1}}. \tag{2.19}$$

As $\{A_1, \ldots, A_{k-2}\}$ is a subgraph of $H(\vec{b})$, we have

$$|A_1 \cap \cdots \cap A_{k-2}| = b_{k-2} + 2b_{k-1} + b_k.$$

Also, for every $B \in \mathcal{B}$, $\{A_1, \ldots, A_{k-2}, B\}$ is a subgraph of $H(\vec{b})$. Thus,

$$|A_1 \cap \cdots \cap A_{k-2} \cap B| = b_{k-1} + b_k.$$

Now,

$$\begin{aligned} |\mathcal{B}| \cdot (b_{k-1} + b_k) &= |\{(x, B) : \ x \in A_1 \cap \cdots \cap A_{k-2}, B \in \mathcal{B}, x \in B\}| \\ &= \sum_{x \in A_1 \cap \cdots \cap A_{k-2}} \deg_{\mathcal{B}}(x). \end{aligned}$$

Let D be the maximum degree of a vertex in F_1. Then, by (2.19),

$$D \cdot (b_{k-2} + 2b_{k-1} + b_k) \geq |\mathcal{B}| \cdot (b_{k-1} + b_k) \geq \frac{1}{4}(b_{k-1} + b_k) \cdot \frac{m}{t^{\frac{k-1}{2}}}. \tag{2.20}$$

Also, note that

$$\frac{b_{k-1} + b_k}{b_{k-2} + 2b_{k-1} + b_k} \geq \frac{b_{k-1}}{b_{k-2} + 2b_{k-1}} \iff b_k(b_{k-2} + b_{k-1}) \geq 0.$$

Therefore (2.20) gives us,

$$D \geq \frac{1}{4} \cdot \frac{b_{k-1}}{b_{k-2} + 2b_{k-1}} \cdot \frac{m}{t^{\frac{k-1}{2}}}. \tag{2.21}$$

Now, we notice that if x is a vertex of degree D, then deleting it from the edges through x gives us a family of uniformity one less than that of F_1. By induction on b_k, this subfamily already contains a $H(b_1, \ldots, b_{k-1}, b_k - 1)$-free family of size $f(D, H(b_1, \ldots, b_{k-1}, b_k - 1))$, which is a natural lower bound to our problem. Therefore,

$$t \geq f(D, H(b_1, \ldots, b_{k-1}, b_k - 1))$$

We now split into two cases.

- **Case I: $k \geq 4$.** Now we use the inductive lower bound given by (2.18):

$$t \geq D^{\frac{1}{kb_k}} \left(\frac{b_{k-1}}{4(b_{k-2} + 2b_{k-1})} \right)^{\frac{1}{k}} \iff D \leq \left(\frac{4(b_{k-2} + 2b_{k-1})}{b_{k-1}} \right)^{b_k} \cdot t^{kb_k}.$$

Combining this bound with (2.21), we get

$$\left(\frac{4(b_{k-2}+2b_{k-1})}{b_{k-1}}\right)^{b_k} \cdot t^{kb_k} \geq \frac{b_{k-1}}{4(b_{k-2}+2b_{k-1})} \cdot \frac{m}{t^{\frac{k-1}{2}}},$$

Which, on invoking $t^{\frac{k-1}{2}} \leq t^k$, leads us to

$$t^{k(b_k+1)} \geq m \left(\frac{b_{k-1}}{4(b_{k-2}+2b_{k-1})}\right)^{b_k+1},$$

finishing off the induction step.

- **Case II:** $k = 3$**.** In this case we use the inductive lower bound in (2.18) of

$$t \geq D^{\frac{1}{b_3+1}} \left(\frac{b_2}{4(b_1+2b_2)}\right)^{\frac{b_3}{b_3+1}} \iff D \leq \left(\frac{4(b_1+2b_2)}{b_2}\right)^{b_3} \cdot t^{b_3+1}.$$

  Again, combining this bound with (2.21), we obtain

$$\left(\frac{4(b_1+2b_2)}{b_2}\right)^{b_3} \cdot t^{b_3+1} \geq \frac{b_2}{4(b_1+2b_2)} \cdot \frac{m}{t}.$$

  This implies $t \geq m^{\frac{1}{b_3+2}} \left(\frac{b_2}{4(b_1+2b_2)}\right)^{\frac{b_3+1}{b_3+2}}$, completing the induction step.

□

2.6 Proof of Theorem 2.2.10

In this section, we prove Theorem 2.2.10. For the proof, we rely upon the incidence structure of Miquelian inversive planes $\mathbf{M}(q)$ of order q. An inversive plane consists of a set of points $\mathcal{P}$ and a set of circles $\mathcal{C}$ satisfying three axioms [16]:

- Any three distinct points are contained in exactly one circle.
- If $P \neq Q$ are points and c is a circle containing P but not Q, then there is a unique circle b through P, Q and satisfying $b \cap c = \{P\}$.

- $\mathcal{P}$ contains at least four points not on the same circle.

Every inversive plane is a 3-$(n^2+1, n+1, 1)$-design for some integer n, which is called its order. An inversive plane is called Miquelian if it satisfies Miquel's theorem [16]. The usefulness of Miquelian inversive planes lies in the fact that their automorphism groups are sharply 3-transitive (cf. pp 274-275, Section 6.4 of [17]). There are a few known constructions of $\mathbf{M}(q)$, one such construction is outlined here. The points of $\mathbf{M}(q)$ are elements of $\mathbb{F}_q^2$ and a special point at infinity, denoted by ∞. The circles are the images of the set $K = \mathbb{F}_q \cup \{\infty\}$ under the permutation group $PGL_2(q^2)$, given by

$$x \mapsto \frac{ax^\alpha + c}{bx^\alpha + d},\ ad - bc \neq 0, \alpha \in \mathrm{Aut}(\mathbb{F}_q^2).$$

For further information on inversive planes and their constructions, the reader is referred to [17, 6, 64].

Now, we prove Theorem 2.2.10.

Proof of Theorem 2.2.10. Recall that for every odd prime power q, we are required to demonstrate a hypergraph on q^2+1 edges with the property that every three edges form an $H(q^2-q-1, q, 1)$. Let $\mathbf{M}(q)$ be a Miquelian inversive plane, with points labeled $\{1, 2, \ldots, q^2+1\}$. Then, we consider the (q^2+q)-graph $F = \{A_1, \ldots, A_{q^2+1}\}$, whose vertex set $V(F)$ is the circles of $\mathbf{M}(q)$, and A_i is the collection of circles containing i. By the inversive plane axiom, any three distinct points have a unique circle through them. It suffices to show that any two distinct points P, Q in $\mathbf{M}(q)$ have $q+1$ distinct circles through them. By 2-transitivity of the Automorphism group, we know that any two points have the same number a_2 of circles through them. Now, for any $P \neq Q$,

$$\begin{aligned}(q^2+1-2)\cdot 1 &= |\{(R, c) : R \text{ is a point}, c \text{ is a circle through } P, Q, R\}| \\ &= a_2 \cdot (q+1-2),\end{aligned}$$

Thus $a_2 = q+1$. So, F is (q^2+q)-uniform, every two edges of F have an intersection of size $q+1$, and every three edges of F have an intersection of size 1. By inclusion-exclusion, they form a $H(q^2-q-1, q, 1)$. □

Now, we prove Corollary 2.2.11.

Proof of Corollary 2.2.11. First, we prove (2.4), which asserts that whenever $b_1 \geq b_2^2 \geq m$ and b_2 is a prime power, $f(m, H_3(b_1, b_2, 1)) = 2$. Initially we start with an inversive plane construction, which gives gives us b_2^2+1 sets such that any three of them are an isomorphic copy of $H(b_2^2-b_2-1, b_2, 1)$. As long as $b_2^2+1 \geq m$, we can take a subgraph of the construction and still obtain m sets satisfying the same property. Also note that as $b_1 \geq b_2^2$, we can create a $H_3(b_1, b_2, 1)$-construction by first creating an inversive plane F, which is a $H_3(b_2^2-b_2-1, b_2, 1)$-construction, and then adding $(b_1-b_2^2+b_2+1)$ new distinct points to each set in F. This proves (2.4).

To prove (2.5), we shall use the result of Baker, Harman and Pintz [2] on the density of primes, which states that for sufficiently large x there is a prime p such that

$$x - x^{0.525} < p < x.$$

Let $g(x)$ be the inverse of $x - x^{0.525}$ for large x. Then, $x < g(p)$. Using monotonicity of g, it can be shown that $g(p) < p + p^{0.529}$ for large p. Thus, for large enough m, there exists a prime p such that

$$p < x < p + p^{0.529}. \tag{2.22}$$

Now, let $b_1 \gg b_2$ and $b_2 \geq m^{0.68}$, as in the hypothesis. From (2.22), we get a prime number p with $p < b_2 < p+p^{0.529}$. Let $F = \{A_1, \cdots, A_{p^2+1}\}$ be the $H_3(p^2-p-1, p, 1)$-construction obtained from Theorem 2.2.10. Note that $m < b_2^{0.68^{-1}} = b_2^{1.4706} < p^2+1$. Let $F' = \{A_1, \cdots, A_m\}$. For every $1 \leq i < j \leq m$, add $b_2 - p$ many new vertices

$v_1^{ij}, \ldots, v_{b_2-p}^{ij}$ to the sets A_i and A_j, i.e, let

$$B_i = A_i \sqcup \bigcup_{j \neq i} \{v_r^{ij} : 1 \leq r \leq b_2 - p\}.$$

Suppose $K = \{B_i : 1 \leq i \leq m\}$. Observe that for every i,

$$|B_i| = p^2 + p + m(b_2 - p),$$

and for every $i \neq j$,

$$|B_i \cap B_j| = p + b_2 - p = b_2.$$

Hence, K is a hypergraph such that any three edges form a $H_3(p^2+p+m(b_2-p), b_2, 1)$. Since

$$\begin{aligned} p^2 + p + m(b_2 - p) &< p^2 + p + p^{1.4706+0.529} \\ &= p^2 + p^{1.9996} + p \\ &< 3b_2^2 \ll b_1, \end{aligned}$$

we can add adequately many new vertices to every edge of K in order to get a hypergraph whose any three edges form a $H_3(b_1, b_2, 1)$. □

2.7 Further Problems and Discussion

We discuss a few problems that are of interest. Of course, the main open question is (2.2), which asks to characterize all sequences of k-edge hypergraphs H_m for which $f(m, H_m)$ is bounded. As we discussed, even the case $k = 3$ turns out to be quite challenging..

Let us focus on the case $k = 3$ and $\vec{\mathbf{b}} = (b_1, b_2, 1)$. The current state of affairs was summarized in Figure 2-3. Observe that all the upper bounds in the lightly shaded regions are actually upper bounds of 2. Therefore, one may ask the following question:

Problem 2.7.1. Characterize all values of (b_1, b_2) such that

$$f(m, H_3(b_1, b_2, 1)) = 2.$$

We cannot solve this problem completely. However, we can derive a necessary condition on b_1, b_2, b_3 for which $f(m, H_3(b_1, b_2, b_3)) = 2$ as follows. Suppose F is a hypergraph with $V(F) = \{1, \cdots, n\}$ such that any three edges of F form a $H_3(b_1, b_2, b_3)$. Let d_i denote the degree of vertex i in F. By double-counting arguments,

$$\begin{aligned}\sum_{i=1}^{n} \binom{d_i}{3} &= \binom{m}{3} b_3, \\ \sum_{i=1}^{n} \binom{d_i}{2} &= \binom{m}{2} (b_2 + b_3), \\ \sum_{i=1}^{n} d_i &= m(b_1 + 2b_2 + b_3).\end{aligned}$$

After algebraic manipulation of these expressions and using the Cauchy-Schwarz inequality $\sum_{i=1}^{n} d_i \cdot \sum_{i=1}^{n} d_i^3 \geq \left(\sum_{i=1}^{n} d_i^2\right)^2$ and large m, we obtain Theorem 2.7.2.

Theorem 2.7.2. *Suppose $f(m, H_3(b_1, b_2, b_3)) = 2$. Then, for large enough m,*

$$b_1 b_3 + \frac{b_1 b_2}{m} + \frac{b_2 b_3}{m} \geq b_2^2.$$

In particular, when $b_3 = 1$,

$$b_1 + \frac{b_1 b_2}{m} \geq b_2^2.$$

Theorem 2.7.2 gives more insight into Figure 2-3. Basically, there are two cases to consider. When b_1 is asymptotically larger than $\frac{b_1 b_2}{m}$, i.e. when $b_2 = o(m)$, this means that $b_1 \geq b_2^2$ is necessary for $f = 2$. When $b_2 \geq m$, this gives us $b_1 \geq m b_2$, which is exactly the construction in Lemma 2.4.3. Further, note that this transition occurs exactly at the intersection of the line $b_1 = m b_2$ and the parabola $b_1 = b_2^2$.

As a further special case of Problem 2.7.1, one can look at $\vec{\mathbf{b}} = (m, b_2, 1)$ where $1 \ll b_2 \ll \sqrt{m}$. We expect this range to be solvable via a construction, since there are constructions for $b_2 = 1$ (Theorem 2.2.7) and $b_2 = \sqrt{m}$ (Theorem 2.2.10). The problem is equivalent to constructing bipartite graphs with certain properties, as stated below.

Problem 2.7.3. Suppose $1 \ll b_2 \ll \sqrt{m}$. Is there a bipartite graph G with parts A, B, such that $|A| = m$, the degree of every vertex in A is asymptotic to m, the size of the common neighborhood of every pair in A is asymptotic to b_2, and every three vertices in A have a unique common neighbor in B?

If such a bipartite graph can be constructed, then we can let $F = \{N_G(u) : u \in A\}$. This hypergraph will testify for $f(m, H(m, b_2, 1)) = 2$. From the proof of Theorem 2.7.2, we know that if such a bipartite graph exists, it cannot be regular from B. A regular construction from B implies equality in the Cauchy-Schwarz inequality, which would imply $b_2 = \Theta(\sqrt{m})$. Therefore if such a graph is constructed, B needs to have vertices of different degrees.

Notice also that if the answer to Problem 2.7.3 is affirmative, then we can shade the small triangle in Figure 2-3 light. This is courtesy of the fact that given any (b_1, b_2) in that region, we can write it as a sum of $(x, y)+(m, z)$, with $x \geq my$. We can then take a $H_3(x, y, 0)$-construction $\{A_1, \ldots, A_m\}$ and a $H_3(m, z, 1)$-construction $\{A'_1, \ldots, A'_m\}$, and merge them together to obtain the $H_3(b_1, b_2, 1)$-construction $\{A_1 \cup A'_1, \ldots, A_m \cup A'_m\}$.

Chapter 3

THE GENERALIZED TURÁN PROBLEM FOR COUNTING TRIANGLES

3.1 Background

For a graph $G = (V(G), E(G))$, let $e(G)$ denote the number of edges of G. Recall that G is F-free if G contains no subgraph isomorphic to H. We emphasize that we do not consider induced subgraphs in this definition.

Given graphs T and H without isolated vertices, the generalized Turán number $\mathrm{ex}(n, T, H)$ is the maximum number of copies of T in an n-vertex graph with no copies of H. The systematic study of $\mathrm{ex}(n, T, H)$ for $T \neq K_2$ was initiated by Alon and Shikhelman [1]. Starting with analyzing $\mathrm{ex}(n, T, H)$ for complete graphs [18, 7, 41] since the 1960's, there has been a lot of recent activity when $(T, H) = (K_3, C_{2k+1})$ and $(T, H) = (C_5, K_3)$ [1, 8, 31, 40, 38]. In [36], the cases $(T, H) = (P_k, K_{2,t})$ and $(T, H) = (C_k, K_{2,t})$ have also been studied and some generic bounds on $\mathrm{ex}(n, T, H)$ are given. See also [50] for a related result about the number of s-cliques in graphs without cycles of length at least k.

Alon and Shikhelman [1] determine all pairs of graphs T, H with $\mathrm{ex}(n, T, H) = \Theta(n^{|V(T)|})$. Further, they prove that if T and H are trees, then there exists an $m(T, H)$ such that $\mathrm{ex}(n, T, H) = \Theta(n^{m(T,H)})$. They also study the problem when H is a tree and T is a bipartite graph, and give several results on $\mathrm{ex}(n, K_t, H)$ for bipartite H. One general result they prove using a theorem of Erdős [20] is that if the chromatic number $\chi(H) > t$, then

$$\mathrm{ex}(n, K_t, H) = \binom{\chi(H) - 1}{t} \left(\frac{n}{\chi(H) - 1} \right)^t + o(n^t).$$

In [51], the error term was determined more precisely. Observe that as a corollary, when $\chi(H) > 3$, we have $\mathrm{ex}(n, K_3, H) = \left(\frac{(\chi(H)-2)(\chi(H)-3)}{24(\chi(H)-1)^2} + o(1) \right) n^3$.

3.2 Our Results

All our results concern $T = K_3$. Since the asymptotic formula of $\mathrm{ex}(n, K_3, H)$ is already very precisely known for $\chi(H) > 3$ and [1] studies the case $\chi(H) = 2$ quite extensively, we consider the wide open case $\chi(H) = 3$. Even within this class, we restrict our attention to a very specific and simple family of 3-chromatic graphs: the suspensions of bipartite graphs. For any graph H, recall that $\widehat{H}$ denotes the suspension $K_1 \vee H$ obtained by adding a new vertex adjacent to every vertex of H. Our contribution consists of lower and upper bounds on $\mathrm{ex}(n, K_3, \widehat{H})$ for $H \in \{K_{a,b}, C_{2k}, P_k\}$.

Given a graph $G = (V, E)$ and a vertex $v \in V$, let $N_G(v) = \{u \in V : uv \in E\}$ denote the neighborhood of v in G. For any subset $X \subseteq V$, let $e(X)$ denote the number of edges in the subgraph $G[X]$ induced by X. Let $t(G)$ denote the number of triangles in G.

If G is $\widehat{H}$-free, then $N_G(v)$ is H-free for every $v \in V$. This implies that

$$t(G) = \frac{1}{3}\sum_{v\in V} e(N_G(v)) \leq \frac{1}{3}\sum_{v\in V} \mathrm{ex}(|N_G(v)|, H) \leq \frac{n}{3} \cdot \mathrm{ex}(n, H).$$

Hence,

$$\mathrm{ex}(n, K_3, \widehat{H}) \leq \frac{n}{3} \cdot \mathrm{ex}(n, H). \tag{3.1}$$

All our results give improvements on (3.1). For our first result $H = K_{a,b}$ and $\widehat{H} = K_{1,a,b}$, where $1 \leq a \leq b$. Here (3.1) combined with the Kövari-Sós-Turán theorem [44], which asserts that $\mathrm{ex}(n, K_{a,b}) = O(n^{2-\frac{1}{a}})$ yields $\mathrm{ex}(n, K_3, K_{1,a,b}) = O(n^{3-\frac{1}{a}})$. We improve this as follows.

Theorem 3.2.1. *For fixed $1 \leq a \leq b$ and $n \to \infty$,*

$$\mathrm{ex}(n, K_3, K_{1,a,b}) = o(n^{3-\frac{1}{a}}). \tag{3.2}$$

As a corollary of (3.2) we obtain $\mathrm{ex}(n, K_3, K_{1,2,2}) = o(n^{5/2})$. We also demonstrate that $\mathrm{ex}(n, K_3, K_{1,2,2}) \geq \Omega(n^2)$ via a construction in Proposition 3.4.1.While we believe that the upper bound is closer to the ground truth, there has been no lower bound on $\mathrm{ex}(n, K_3, K_{1,2,2})$ till date which is of an order of magnitude more than $\Omega(n^2)$, to the best knowledge of the author.

Our next result concerns $H = C_{2k}$. Here (3.1) together with the classical bound $\mathrm{ex}(n, C_{2k}) = O(n^{1+\frac{1}{k}})$ of Bondy-Simonovits [9] yields $\mathrm{ex}(n, K_3, \widehat{C}_{2k}) = O(n^{2+\frac{1}{k}})$.

Theorem 3.2.2. *For fixed $k \geq 2$ and $n \to \infty$,*

$$ex(n, K_3, \widehat{C}_{2k}) = o(n^{2+\frac{1}{k}}). \tag{3.3}$$

The lower bound construction for $\mathrm{ex}(n, K_3, K_{1,2,2})$ in Proposition 3.4.1 also yields $\mathrm{ex}(n, K_3, \widehat{C}_{2k}) = \Omega(n^2)$.

Our final results concern $\mathrm{ex}(n, K_3, \widehat{P}_k)$ for $k \geq 3$. We begin with a simple proposition.

Proposition 3.2.3. *Let $n \geq k \geq 3$. Then*

$$\left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8} \leq ex(n, K_3, \widehat{P}_k) \leq \frac{k-1}{12} \cdot n^2 + \frac{(k-1)^2}{12} \cdot n, \tag{3.4}$$

where the lower bound holds when n is a multiple of $4\lfloor \frac{k-1}{2} \rfloor$.

We believe that the lower bound above is asymptotically tight for all fixed $k \geq 3$ and prove this for the first three cases $k = 3$, 4 and 5.

Theorem 3.2.4. *For $k = 3$, 4 and 5,*

$$ex(n, K_3, \widehat{P}_k) = \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8} + o(n^2). \tag{3.5}$$

When $k = 3$ or $k = 5$, the error term can be improved to $O(n)$.

We think that (3.5) holds for all values of k. Assuming that the number of edges of high codegree in a $\widehat{P}_k$-free graph is small (Conjecture 3.6.1), we also deduce the result for general k, as demonstrated in Proposition 3.2.5.

Proposition 3.2.5. *Let n be sufficiently large, and suppose G is any n-vertex $\widehat{P}_k$-free graph. Let E_h denote the set of edges which lie in more than $\frac{k-1}{2}$ triangles in G. If $|E_h| = o(n^2)$, then $t(G) \leq \frac{k-1}{16} \cdot n^2 + o(n^2)$.*

In Section 2, we present some preliminary results that we use in our proofs. In Section 3 we prove Theorem 3.2.1, in Section 4 we prove Theorem 3.2.2 and in Section 5 we prove Proposition 3.2.3, Proposition 3.2.5 and Theorem 3.2.4.

3.3 Preliminaries

In this section, we describe some preliminary tools that will be used in our proofs. The most important tool for proving Theorems 3.2.1 and 3.2.2 is the triangle removal lemma [13, 33, 65]. The specific form that we shall be using appears as Theorem 2.1 in [13].

Lemma 3.3.1 (Ruzsa-Szemerédi [65])**.** *Suppose $\epsilon > 0$. Let $\delta = \delta(\epsilon)$ be such that $\frac{1}{\delta}$ is a tower of twos of height $684\log(\frac{1}{\epsilon})$. If G is a graph on n vertices with at least ϵn^2 edge-disjoint triangles, then G contains at least δn^3 triangles.*

Recall that $t(G)$ is the number of triangles in G.

Lemma 3.3.2 (Nordhaus-Stewart [60])**.** *For any graph G on n vertices,*

$$t(G) \geq \frac{e(G)}{3n} \cdot \left(4e(G) - n^2\right).$$

A k-uniform hypergraph $H = (V(H), E(H))$, consists of a vertex set $V(H)$ and edge set $E(H) \subseteq \binom{V(H)}{k}$. We write $e(H) = |E(H)|$. A subset $X \subseteq V(H)$ of vertices is an independent set if $e \not\subseteq X$ for any $e \in E(H)$. The independence number of H, denoted by $\alpha(H)$, is the largest size of an independent set in H. Recall Lemma 2.5.3 from Chapter 3, which states that for an r-uniform hypergraph $H = (V, E)$ n vertices, if $d = \frac{re(H)}{n}$ denotes the average degree of H, then

$$\alpha(H) \geq \frac{r-1}{r} \cdot \left(\frac{n}{d^{\frac{1}{r-1}}}\right).$$

Finally, we require a result that is a direct consequence of the proof of the Erdős-Gallai theorem [22] for cycles, which states that every graph with average degree at least k contains a cycle of length at least $k+1$. Recall that a chord in a cycle is an edge between any two non-adjacent vertices of the cycle.

Lemma 3.3.3 (Erdős-Gallai [22])**.** *Let $k \geq 3$ be an integer. If G is a graph of average degree at least k, then G contains a cycle of length at least $k+1$ with a chord. In particular, G also contains a path of length at least k.* □

3.4 Suspension of complete bipartite graphs

Our goal in this short section is to prove Theorem 3.2.1 and give a lower bound on $\mathrm{ex}(n, K_3, K_{1,2,2})$ via Proposition 3.4.1. Given a graph G and $X \subseteq V(G)$, let $N_G(X) = \bigcap_{v \in X} N_G(v)$ denote the common neighborhood of all vertices from X.

Proof of Theorem 3.2.1. Recall that we wish to prove that when $1 \leq a \leq b$, we have $\mathrm{ex}(n, K_3, K_{1,a,b}) = o(n^{3-\frac{1}{a}})$. Let n be sufficiently large, and G be an n-vertex graph which is $K_{1,a,b}$-free. By the Kövari-Sós-Turán theorem [44] and (3.1), we know that there exists $c = c_b$ such that

$$t(G) < c_b n^{3-\frac{1}{a}}. \tag{3.6}$$

Now, suppose $\epsilon > 0$ is fixed. Assume that n is sufficiently large, and that G is a $K_{1,a,b}$-free graph with $t(G) \geq \epsilon n^{3-\frac{1}{a}}$. Construct an a-uniform hypergraph H whose vertices are the triangles of G, and let $\{T_1, \ldots, T_a\}$ be an edge of H if the triangles $T_1, \ldots, T_a$ all share a common edge in G. Then

$$e(H) = \sum_{\{v_1,\ldots,v_a\} \in \binom{V(G)}{a}} e(N_G(\{v_1, \ldots, v_a\})).$$

As G is $K_{1,a,b}$-free, $N_G(\{v_1, \ldots, v_a\})$ has no vertex of degree at least b. This implies that

$$e(N_G\{v_1, \ldots, v_a\}) < \frac{b}{2} \cdot |N_G(\{v_1, \ldots, v_a\})|.$$

Consequently,

$$\begin{aligned} e(H) &< \sum_{\{v_1,\ldots,v_a\} \in \binom{[n]}{a}} \frac{b}{2} \cdot |N_G(v_1, \ldots, v_a)| \\ &= \frac{b}{2} \sum_{v \in V(G)} \binom{\deg(v)}{a} \\ &< b \sum_{v \in V(G)} \deg(v)^a \\ &< b \cdot n^{a+1}. \end{aligned}$$

This implies that $d(H) < \frac{ab \cdot n^{a+1}}{t(G)}$. Using Lemma 2.5.3,

$$\alpha(H) > \frac{k-1}{k} \cdot \frac{t(G)}{\left(\frac{ab \cdot n^{a+1}}{t(G)}\right)^{\frac{1}{a-1}}} = c \cdot t(G)^{1+\frac{1}{a-1}} \cdot n^{-\frac{a+1}{a-1}},$$

where $c = (k-1)/(k(ab)^{1/(a-1)})$. Recalling that $t(G) \geq \epsilon n^{3-\frac{1}{a}}$ and letting $\epsilon' = c \cdot \epsilon^{1+\frac{1}{a-1}}$, we obtain

$$\alpha(H) > \epsilon' n^{\frac{a}{a-1} \cdot \frac{3a-1}{a}} \cdot n^{-\frac{a+1}{a-1}} = \epsilon' n^2.$$

Let I be a maximum independent set of H. Create an auxiliary graph H' with vertex set I, and join two vertices of H' iff the triangles corresponding to them share an edge. Every triangle from I can be adjacent to at most $3(a-1)$ other triangles from I. Therefore, $\deg_{H'}(i) < 3a$ for every $i \in I$, and hence by Lemma 2.5.3 H' has an independent set of size at least $\frac{|I|}{6a} > \frac{\epsilon' n^2}{6a}$. The triangles corresponding to this independent set are edge-disjoint. Therefore $t(G) \geq \delta n^3$ where $\delta = \delta(\frac{\epsilon'}{6a})$ is obtained from Lemma 3.3.1. However $t(G) < c_b n^{3-\frac{1}{a}}$ by (3.6) and this implies that $\delta n^3 \leq \frac{c_b}{3} \cdot n^{3-\frac{1}{a}}$, a contradiction for sufficiently large n. □

Plugging in $a = b = 2$ in Theorem 3.2.1, we get the bound $\text{ex}(n, K_3, K_{1,2,2}) = o(n^{5/2})$. We now describe two lower bound constructions for $\text{ex}(n, K_3, K_{1,2,2})$, one of which is of the order $\frac{n^2}{4} + o(n^2)$, and another of the order $\frac{n^2}{6} + o(n^2)$.

Proposition 3.4.1. *When n is a multiple of 4, there is a $K_{1,2,2}$-free graph on n vertices and $\frac{n^2}{4}$ edges which is edge-maximal.*

Proof of Proposition 3.4.1. Let $H_n = (A, B)$ be the complete bipartite graph with $|A| = |B| = \frac{n}{2}$, with additional edges in both the parts such that $H_n[A]$ and $H_n[B]$ are matchings of size $\frac{n}{4}$. Observe that the neighborhood of every vertex in H_n consists of $\frac{n}{4}$ edge-disjoint triangles sharing a common vertex, and hence is C_4-free. Thus H_n is $K_{1,2,2}$-free. On the other hand, every triangle of H_n either has an edge inside A or an edge inside B, implying

$$t(H_n) = e(A) \cdot |B| + e(B) \cdot |A| = 2 \cdot \frac{n}{4} \cdot \frac{n}{2} = \frac{n^2}{4}.$$

This proves that whenever $4 \mid n$, $\mathrm{ex}(n, K_3, K_{1,2,2}) \geq \frac{n^2}{4}$. □

3.5 Suspension of even cycles

Our goal in this section is to prove Theorem 3.2.2. Before proceeding with the proof, we prove Lemma 3.5.2 which gives an upper bound on the number of paths of length k in a C_{2k}-free graph. The main idea behind the lemma is the technique used in [70, 62, 71] to prove upper bounds on $\mathrm{ex}(n, C_{2k})$ by analyzing the breadth-first search tree from any vertex.

Given a graph G and a vertex $r \in V(G)$, a breadth-first search tree T of G rooted at r is constructed as follows. Let $L_0 = \{r\}$. For $i \geq 1$, let $L_i \subseteq V(G)$ be the set of all vertices in $V(G)$ which are at distance i from vertex r. The vertex subset L_i is called the i'th level of T. The tree T consists of vertex set $V(G)$ and only the edges of G between levels L_i and L_{i+1}, $i \geq 0$.

For $i \geq 0$, let $G[L_i]$ be the subgraph of G induced by L_i, and let $G[L_i, L_{i+1}]$ be the bipartite subgraph of G with parts (L_i, L_{i+1}) and edges exactly the edges of G that have one endpoint in L_i and another in L_{i+1}.

We now quote Lemma 3.5 from [71] in the form that we shall be using.

Lemma 3.5.1 (Verstraëte [71])**.** *Let T be a breadth-first search tree in a graph G, with levels $L_0, L_1, \ldots$. Suppose $G[L_i]$ or $G[L_i, L_{i+1}]$ has a cycle of length k with a chord, respectively. Then, for some $m \leq i$, G contains cycles $C_{2m+1}, C_{2m+2}, \ldots, C_{2m+k-1}$, or cycles $C_{2m+2}, C_{2m+4}, \ldots, C_{2m+\ell}$ respectively, where ℓ is the largest even integer less than k.*

Let $p_k(G)$ denote the number of paths of length k in a graph G. Here each subgraph isomorphic to P_k is counted twice, once for each ordering of its vertices along the path.

Lemma 3.5.2. *Let $k \geq 2$ be an integer, $0 < \epsilon < 1$ and $n > (20k/\epsilon)^k$. Let F be a C_{2k}-free graph on n vertices with minimum degree at least $\epsilon n^{1/k}$. Then*

$$p_k(F) \leq \left(\frac{2k}{\epsilon}\right)^{(k-1)k} n^2.$$

Proof of Lemma 3.5.2. We first prove that F has bounded maximum degree. Suppose for contradiction that there exists $v \in V(F)$ with

$$\deg(v) \geq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k}.$$

Consider the breadth-first search tree of F starting at v. For $i \geq 0$, let L_i be the ith level of this breadth-first search tree. By assumption, $|L_1| \geq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k}$. Denote by $e(L_i, L_{i+1})$ the number of edges in $G[L_i, L_{i+1}]$. Let us prove that for every $1 \leq i < k$,

$$e(L_i) \leq (k-1)|L_i| \quad \text{and} \quad e(L_i, L_{i+1}) \leq (k-1)(|L_i| + |L_{i+1}|). \tag{3.7}$$

Indeed, if $e(L_i) > (k-1)|L_i|$, then by Lemma 3.3.3, $F[L_i]$ contains a cycle of length ℓ with a chord, where $\ell \geq 2k-1$. Now, apply Lemma 3.5.1 to obtain an integer $m \leq i$ such that F contains cycles of lengths $2m+1, 2m+2, \ldots, 2m+\ell-1$. Since $\ell \geq 2k-1$ and $1 \leq m < k$, we have $2m+1 \leq 2k \leq 2m+\ell-1$. Then F contains a C_{2k}, contradiction. Similarly, if $e(L_i, L_{i+1}) > (k-1)(|L_i|+|L_{i+1}|)$, then Lemma 3.3.3 gives us a cycle of length ℓ in $F[L_i, L_{i+1}]$ where $\ell \geq 2k-1$. Then Lemma 3.5.1 gives an integer $m \leq i$ such that F contains cycles of lengths $2m+2, 2m+4, \ldots, 2m+\ell$. As $\ell \geq 2k-1$ and $1 \leq m < k$, we have $2m+2 \leq 2k \leq 2m+2k-1$. This implies that F contains a C_{2k}, again a contradiction.

Claim 3.5.3. For every $i \geq 0$,

$$|L_{i+1}| \geq \frac{\epsilon n^{1/k}}{2k} \cdot |L_i|. \tag{3.8}$$

Proof of Claim 3.5.3. We use induction on i. Note that $|L_0| = 1$ and

$$|L_1| = \deg(v) \geq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k} > \frac{\epsilon n^{1/k}}{2k} = \frac{\epsilon n^{1/k}}{2k}|L_0|.$$

Moreover, for $i \geq 1$ and any vertex $v \in L_i$, $N_G(v) \subseteq L_{i-1} \cup L_i \cup L_{i+1}$. Thus, (3.7) implies

$$\begin{aligned} k(|L_i| + |L_{i+1}|) + 2k|L_i| + k(|L_i| + |L_{i-1}|) &> e(L_i, L_{i+1}) + 2e(L_i) + e(L_i, L_{i-1}) \\ &= \sum_{v \in L_i} \deg(v) \\ &\geq \epsilon n^{1/k} \cdot |L_i|. \end{aligned}$$

Consequently,

$$|L_{i+1}| > \left(\frac{\epsilon n^{1/k}}{k} - 4\right) \cdot |L_i| - |L_{i-1}|. \tag{3.9}$$

By the induction hypothesis we may assume that $|L_{i-1}| \leq \frac{2k}{\epsilon n^{1/k}} \cdot |L_i|$. Thus, (3.9) implies

$$|L_{i+1}| > \left(\frac{\epsilon n^{1/k}}{k} - 4 - \frac{2k}{\epsilon n^{1/k}}\right) \cdot |L_i| > \frac{\epsilon n^{1/k}}{2k} \cdot |L_i|$$

since $n > (20k/\epsilon)^k$. This finishes the proof of Claim 3.5.3.

Now, by applying Claim 3.5.3 iteratively, we obtain

$$|L_k| \geq \left(\frac{\epsilon n^{1/k}}{2k}\right)^{k-1} \cdot |L_1| \geq \left(\frac{\epsilon n^{1/k}}{2k}\right)^{k-1} \cdot \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k} = n,$$

a contradiction. Thus, $\deg(v) \leq \left(\frac{2k}{\epsilon}\right)^{k-1} \cdot n^{1/k}$ for every $v \in V(F)$. Therefore, if $\Delta(F)$ is the maximum degree of F,

$$p_k(F) \leq n \cdot \Delta(F)^k \leq \left(\frac{2k}{\epsilon}\right)^{k(k-1)} \cdot n^2,$$

as desired. □

For a graph G and edge $uv \in E(G)$, the codegree of uv is $\deg_G(u, v) = |N_G(\{u, v\})|$.

Proof of Theorem 3.2.2. Fix $\epsilon > 0$ and let n be sufficiently large. Suppose $G = (V, E)$ is a graph on n vertices satisfying $t(G) \geq \epsilon n^{2+\frac{1}{k}}$. We wish to show that G contains a copy of $\widehat{C}_{2k}$. Suppose, on the contrary, that G is $\widehat{C}_{2k}$-free. First, we iteratively delete edges of G with codegree less than $\frac{\epsilon n^{1/k}}{10}$ in the current graph, until there are no such edges left. Since we delete fewer than $e(G) \cdot \frac{\epsilon n^{1/k}}{10}$ triangles, we are left with a graph G' satisfying

$$t(G') > t(G) - e(G) \cdot \frac{\epsilon n^{1/k}}{10} > \epsilon n^{2+\frac{1}{k}} - \frac{\epsilon n^{2+\frac{1}{k}}}{10} \geq \frac{9\epsilon}{10} \cdot n^{2+\frac{1}{k}},$$

and $\deg_{G'}(u, v) \geq \frac{\epsilon n^{1/k}}{10}$ for every $uv \in E(G')$.

Next, create an auxiliary graph H whose vertices are the triangles of G', and two vertices of H are adjacent iff their corresponding triangles share a common edge. By Lemma 2.5.3, $\alpha(H) > \frac{t(G')}{2d(H)}$. Let

$$\gamma := \frac{(\epsilon/20)^{k^2}}{k^{(k-1)k}} > 0.$$

If $\frac{t(G')}{2d(H)} > \frac{\gamma}{2}n^2$, then this gives us $\frac{\gamma}{2}n^2$ edge-disjoint triangles in G, and this implies that $t(G) > \delta n^3$ where $\delta = \delta(\frac{\gamma}{2})$ from Lemma 3.3.1. However, we also have $t(G) < c_k n^{2+\frac{1}{k}}$ by (3.1) and this is a contradiction since n is sufficiently large. Therefore, we may assume $\frac{t(G')}{d(H)} \leq \gamma n^2$ and this implies

$$d(H) \geq \frac{t(G')}{\gamma n^2} \geq \frac{9\epsilon}{10\gamma} \cdot n^{1/k}$$

and hence

$$e(H) \geq \frac{9\epsilon n^{1/k}}{20\gamma} \cdot t(G') \geq \frac{81\epsilon}{200\gamma} \cdot n^{2+\frac{2}{k}}.$$

Let us now bound X, the number of copies of $\widehat{P}_k$ in G' in two different ways. For every $v \in V(G')$, let G'_v denote the subgraph of G' induced by $N_{G'}(v)$. Let $\delta(F)$ denote the minimum degree of F for any graph F. By the assumption on the minimum codegree of edges in G', $\delta(G'_v) \geq \frac{\epsilon n^{1/k}}{10}$. Hence applying Lemma 3.5.2 with ϵ replaced by $\frac{\epsilon}{10}$,

$$\begin{aligned} X \leq \sum_{v \in V(G')} p_k(G'_v) &\leq n \cdot \left(\frac{20k}{\epsilon}\right)^{k(k-1)} \cdot n^2 \\ &= \left(\frac{20k}{\epsilon}\right)^{k(k-1)} \cdot n^3. \end{aligned} \tag{3.10}$$

On the other hand, we can first fix two adjacent triangles in G' and then keep growing it to a $\widehat{P}_k$ by using the minimum codegree condition of G'. Since $\delta(H) \geq \frac{\epsilon n^{1/k}}{10}$, this implies that for large enough n,

$$\begin{aligned} X &\geq \frac{1}{2} e(H) \cdot (\delta(H) - 2) \cdot (\delta(H) - 3) \cdots (\delta(H) - k + 1) \\ &\geq \frac{1}{2} e(H) \cdot (\delta(H) - k)^{k-2} \\ &\geq \frac{81\epsilon}{400\gamma} \cdot n^{2+\frac{2}{k}} \cdot \left(\frac{\epsilon n^{1/k}}{20}\right)^{k-2} \\ &= \frac{81\epsilon^{k-1}}{20^k \cdot \gamma} \cdot n^3. \end{aligned} \tag{3.11}$$

The factor $\frac{1}{2}$ in (3.11) is to balance out over-counting the same $\widehat{P}_k$ from its two ends. Comparing (3.10) and (3.11), we obtain

$$\left(\frac{20k}{\epsilon}\right)^{k(k-1)} \geq \frac{81\epsilon^{k-1}}{20^k \cdot \gamma},$$

implying

$$\gamma \geq \frac{81\epsilon^{k^2}}{20^{k^2} \cdot k^{k(k-1)}} = 81 \cdot \frac{(\epsilon/20)^{k^2}}{k^{k(k-1)}} > 81\gamma,$$

a contradiction. This completes the proof of Theorem 3.2.2. □

3.6 Suspension of paths

In this section, we prove Proposition 3.2.3, Proposition 3.2.5 and Theorem 3.2.4.

3.6.1 Proof of Proposition 3.2.3

First, we show the upper bound in (3.4). Let $k \geq 3$ be fixed, and let G be a graph on n vertices which is $\widehat{P}_k$-free. We need to show that $t(G) \leq \frac{(k-1)}{12} \cdot n^2 + \frac{(k-1)^2}{12} \cdot n$.

Note that the neighborhood of every vertex $v \in V(G)$ is P_k-free. Thus by Lemma 3.3.3, the average degree of the subgraph of G induced by $N_G(v)$ is at most $k-1$. Hence,

$$e(N_G(v)) \leq \frac{k-1}{2} \cdot \deg_G(v).$$

Summing up this inequality over all vertices $v \in V(G)$,

$$3t(G) = \sum_{v \in V(G)} e(N(v)) \leq \frac{k-1}{2} \cdot 2e(G) = (k-1)e(G),$$

giving us

$$t(G) \leq \frac{k-1}{3} \cdot e(G). \tag{3.12}$$

This, in conjunction with Lemma 3.3.2, gives us

$$\frac{k-1}{3} \cdot e(G) \geq t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

which simplifies to

$$e(G) \leq \frac{n^2}{4} + \frac{(k-1)n}{4}.$$

The conclusion of the upper bound follows from plugging this inequality back into (3.12).

Now we prove the lower bound in (3.4). Let n be a multiple of $4\left\lfloor \frac{k-1}{2} \right\rfloor$. We shall construct a $\widehat{P}_k$-free graph $F_{n,k}$ on n vertices with $t(F_{n,k}) \geq \left\lfloor \frac{k-1}{2} \right\rfloor \cdot \frac{n^2}{8}$.

Let $F_{n,k} = (A, B)$ be the complete bipartite graph with parts A, B with $|A| = |B| = \frac{n}{2}$, with additional edges in A such that $F_{n,k}[A]$ is a disjoint union of $K_{\left\lfloor \frac{k-1}{2} \right\rfloor, \left\lfloor \frac{k-1}{2} \right\rfloor}$. Then,

$$e(A) = \left\lfloor \tfrac{k-1}{2} \right\rfloor^2 \cdot \frac{n}{4\left\lfloor \frac{k-1}{2} \right\rfloor} = \left\lfloor \tfrac{k-1}{2} \right\rfloor \cdot \frac{n}{4}.$$

Every triangle of $F_{n,k}$ consists of an edge from $F_{n,k}[A]$ and a vertex from B. Hence,

$$t(F_{n,k}) = \left\lfloor \tfrac{k-1}{2} \right\rfloor \cdot \tfrac{n}{4} \cdot \tfrac{n}{2} = \left\lfloor \tfrac{k-1}{2} \right\rfloor \cdot \tfrac{n^2}{8}.$$

Further, $F_{n,k}$ is $\widehat{P}_k$-free, since the neighborhood of every vertex in B is a disjoint union of $K_{\lfloor \frac{k-1}{2} \rfloor, \lfloor \frac{k-1}{2} \rfloor}$, and the neighborhood of every vertex in A is isomorphic to $K_{\lfloor \frac{k-1}{2} \rfloor, \frac{n}{2}}$. □

3.6.2 Proof of Proposition 3.2.5

Our main goal in this section is to prove the following result, modulo Conjecture 3.6.1.

$$\mathrm{ex}(n, K_3, \hat{P}_k) \leq \frac{k-1}{16} \cdot n^2 + o(n^2).$$

The proof structure is as follows. First, we will show that for any $\hat{P}_k$-free graph G with $t(G)$ triangles and $e(G)$ edges, we have $t(G) \leq \frac{k-1}{2} \cdot \frac{e(G)}{2} + o(n^2)$. Note that this implies the above theorem by virtue of Lemma 3.3.2.

In order to prove this, we first divide $E(G)$ into two sets: E_h (the heavy edges) and E_ℓ (the light edges). For $e \in E(G)$, let $w(e)$ denote the codegree of e. Define

$$E_h = \left\{ e : w(e) > \frac{k-1}{2} \right\}, E_\ell = \left\{ e : w(e) \leq \frac{k-1}{2} \right\},$$

and for any $v \in V(G)$, let

$$H(v) = \{w \in N(v) : vw \in E_h\}, L(v) = \{w \in N(v) : vw \in E_\ell\}.$$

We conjecture the following result about the number of heavy edges:

Conjecture 3.6.1. $|E_h| = o(n^2)$.

The proof would continue as follows: let t_0, t_1, t_2 and t_3 be the number of K_3's in G with $0, 1, 2$ and 3 heavy edges, respectively. By a counting argument we show that

$6t_0 + 4t_1 + 3t_2 + 3t_3 \leq (k-1)e$. By triangle removal, we observe that $t_0 = o(n^2)$, and finally we assert that $t_2 + t_3 \leq |E_h| = o(n^2)$ from assumption.

We shall prove each of the above statements in different claims.

Claim 3.6.2. For t_0, t_1, t_2, t_3 as above,

$$6t_0 + 4t_1 + 3t_2 + 3t_3 \leq (k-1)e(G).$$

Proof. This follows from counting the sum

$$\sum_{v \in V(G)} \left(\sum_{w \in L(v)} \deg_{N(v)} w + e(H(v)) \right)$$

in two ways. Note that this sum counts every light triangle 6 times, every triangle with one heavy edge twice in the first sum and twice in the second, and so on. Thus this sum is exactly equal to $6t_0 + 4t_1 + 3t_2 + 3t_3$.

On the other hand, observe that $\deg_{N(v)} w \leq \frac{k-1}{2}$ since vw is a light edge. Plus, $e(H(v)) \leq \frac{k-1}{2}|H(v)|$ by Lemma 3.5.2. Thus, we obtain

$$\begin{aligned} 6t_0 + 4t_1 + 3t_2 + 3t_3 &\leq \sum_{v \in V(G)} \left(\frac{k-1}{2}|L(v)| + \frac{k-1}{2}|H(v)| \right) \\ &= \frac{k-1}{2} \sum_{v \in V(G)} \deg v \\ &= (k-1)e(G). \end{aligned}$$

This finishes the proof of the lemma. □

Now, we shall show that the main term in the above sum is $4t_1$, via bounding the other terms to $o(n^2)$.

Claim 3.6.3. $t_0 = o(n^2)$.

Proof. Since t_0 counts the number of triangles which are light, suppose $t_0 \geq \epsilon n^2$. Note that every light triangle can be adjacent to at most $\frac{3(k-1)}{2}$ other triangles, thus we get a set of $\frac{2\epsilon}{3(k-1)} \cdot n^2$ many edge-disjoint triangles in G. By triangle removal lemma, this implies that G had $\geq \delta n^3$ triangles to start with. This is a contradiction to Theorem 3.2.3. □

Claim 3.6.4. $t_2 + t_3 \leq (k-1)|E_h|$.

Proof. This is a simple counting argument. Note that $e(H(v)) \leq \frac{k-1}{2}|H(v)|$ for every $v \in V(G)$. Thus,

$$\begin{aligned} t_2 + 3t_3 &= \sum_{v \in V(G)} e(H(v)) \\ &\leq \sum_{v \in V(G)} \frac{k-1}{2}|H(v)| \\ &= (k-1)|E_h|, \end{aligned}$$

implying the assertion of the claim. □

Finally, we deduce via the above claims that $4t(G) \leq (k-1)e(G) + o(n^2)$. From Lemma 3.3.2, $t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2)$, giving us $t(G) \leq \frac{k-1}{16}n^2 + o(n^2)$, as required.

3.6.3 Proof of Theorem 3.2.4

We will use some ideas from [31], and define the concepts of *triangle-connectivity* and *blocks*. In what follows, a triangle T in a graph G is a set of three edges $\{ab, bc, ca\}$ that form a K_3 in G. Subsequently, we shall denote such a triangle simply as abc.

Definition 3.6.5 (Triangle-connectivity)**.** Given a graph G and two distinct edges $e, e' \in E(G)$, say that e and e' are *triangle-connected* if there is a sequence of triangles $\{T_1, \ldots, T_k\}$ of G, such that $e \in T_1$, $e' \in T_k$, and T_i and T_{i+1} share a common edge for every $1 \leq i \leq k-1$. A subgraph $H \subseteq G$ is *triangle-connected* if e and e' are triangle-connected for every two distinct $e, e' \in E(H)$.

It is straightforward to check that triangle-connectivity is an equivalence relation on $E(G)$ (assuming reflexivity as part of the definition).

Definition 3.6.6 (Triangle block)**.** A *triangle block*, or simply a *block* in a graph G is a subgraph H whose edges form an equivalence class of the *triangle-connectivity* relation on $E(G)$.

In other words, a subgraph $H \subseteq G$ is a triangle block if it is edge-maximally triangle-connected. By definition, the triangle blocks of a graph G are edge-disjoint.

Proof of Theorem 3.2.4 for $\mathbf{k = 3}$.

Suppose G is a graph on n vertices which is $\widehat{P}_3$-free. We will prove using induction on n, that

$$t(G) < \frac{n^2}{8} + 3n. \tag{3.13}$$

This inequality is true for $n = 3$ as $t(G) \leq 1$ for any graph G on 3 vertices. Now, fix an $n > 3$ and a graph G on n vertices which is $\widehat{P}_3$-free. Assume that (3.13) holds for $\widehat{P}_3$-free graphs on less than n vertices.

We may assume without loss of generality that every edge of G lies in a triangle, otherwise we may delete it from G without changing $t(G)$. For a vertex $v \in V(G)$, let $t(v) = t_G(v)$ denote the number of triangles in G containing v. By definition, $t_G(v) = e(N_G(v))$.

We first prove that if G has a copy of K_4, then $t(G) < \frac{n^2}{8} + 3n$, hence completing the induction step.

Suppose G has a copy of K_4 with vertices labeled a_1, a_2, a_3, a_4. Let $X = V(G) \setminus \{a_1, a_2, a_3, a_4\}$, and $A_i = N_G(a_i) \cap X$, $i = 1, \ldots, 4$. If $x \in A_1 \cap A_2$, then we can find a $\widehat{P}_3$ formed by a_1, x, a_2, a_3, a_4 in the neighborhood of a_1. Thus, $A_1 \cap A_2 = \varnothing$, and by symmetry the A_i's are mutually disjoint. Hence, $|A_1| + |A_2| + |A_3| + |A_4| \leq |X| = n - 4$. This implies that one of the A_i's has size $\leq \frac{n-4}{4}$. Using Lemma 3.3.3 in the neighborhood of a_i,

$$t(a_i) = 3 + e(A_i) \leq 3 + |A_i| \leq 3 + \frac{n-4}{4} = \frac{n+8}{4}.$$

Now let $G' = G - a_i$. As G was $\widehat{P}_3$-free, so is G'. Hence by the induction hypothesis,

$$t(G') < \frac{(n-1)^2}{8} + 3(n-1).$$

This implies,

$$t(G) = t(G') + t(a_i) < \frac{(n-1)^2}{8} + 3(n-1) + \frac{n+8}{4} < \frac{n^2}{8} + 3n,$$

as desired. We may now assume that G is K_4-free.

Let B_s denote the *book graph* on $s+2$ vertices, consisting of s triangles all sharing a common edge.

Claim 3.6.7. Every triangle block of G is isomorphic to B_s for some $s \geq 1$.

Proof of Claim 3.6.7. Let $H \subseteq G$ be an arbitrary triangle block. If H contains only one or two triangles, it is isomorphic to B_1 or B_2. Suppose H contains at least three triangles. Let two of them be abx_1 and abx_2 (Figure 3-1). If another triangle is of the form ax_1y for some $y \in V(H)$, then there are two possible cases. If $y \neq x_2$, then $N_H(a)$ contains the 3-path x_2bx_1y. Otherwise, if $y = x_2$, then the vertices a, b, x_1, x_2 create a K_4. Similarly, no triangle contains any of the edges bx_1, ax_2, bx_2. Therefore all triangles in H contain ab and $H \cong B_s$ for some $s \geq 1$.

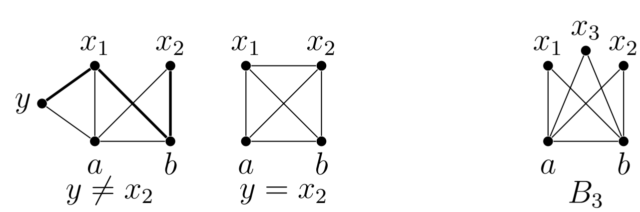

Figure 3-1: (left): third triangle on ax_1, (right): third triangle on ab

Claim 3.6.7 implies that G comprises r edge-disjoint blocks isomorphic to books for some $r \geq 1$. Let the blocks of G be isomorphic to $B_{s_1}, \ldots, B_{s_r}$, where $s_1, \ldots, s_r \geq 1$.

Then,

$$t(G) = s_1 + \cdots + s_r \quad \text{and} \quad e(G) = 2(s_1 + \cdots + s_r) + r = 2t(G) + r.$$

Hence, $t(G) < e(G)/2$. Finally, we apply Lemma 3.3.2 on G to obtain

$$\frac{e(G)}{2} > t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

implying

$$e(G) < \frac{n^2}{4} + \frac{3n}{8}.$$

Therefore $t(G) < \frac{n^2}{8} + \frac{3n}{16} < \frac{n^2}{8} + 3n$, completing the induction step. □

Proof of Theorem 3.2.4 for k = 4.

Suppose $\epsilon > 0$ and n is sufficiently large. Let G be any graph on n vertices which is $\widehat{P}_4$-free, such that

$$t(G) \geq \frac{n^2}{8} + 14\epsilon n^2.$$

The very first step of the proof is to remove copies of K_4 and $K_{1,2,2}$ from G while still maintaining $t(G) \geq \frac{n^2}{8} + \epsilon n^2$. We achieve this by means of the triangle removal lemma. First, we make an observation which follows immediately from Lemma 3.3.1, (3.1) and Lemma 3.3.3 for large n.

$$G \text{ cannot have } \epsilon n^2 \text{ edge-disjoint triangles.} \tag{3.14}$$

Without loss of generality, we may assume that every edge of G is contained in a triangle. We shall use (3.14) to remove all copies of the following six graphs in this order: K_5; K_5^-; K_4; $K_{2,2,2}$; $Q_{3,2} = \overline{K_2} \vee P_3$; and $K_{1,2,2}$ (Figure 3-2).

- **Step 1:** Cleaning K_5's.

  If G contains a K_5 with vertices a_1, a_2, a_3, a_4, a_5, then it has to be a block by

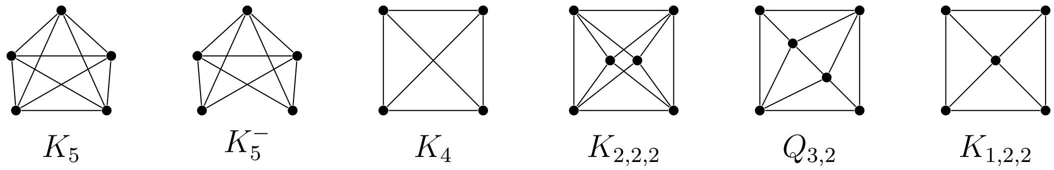

Figure 3-2: The graphs K_5, K_5^-, K_4, $K_{2,2,2}$, $Q_{3,2}$ and $K_{1,2,2}$.

itself. This is because if there is a vertex $x \neq a_i$ with $xa_1, xa_2 \in E(G)$, then $N_G(a_1)$ contains the path $a_5a_4a_3a_2x$, contradiction. Hence, all the K_5's in G are edge-disjoint.

If G has more than ϵn^2 copies of K_5, then by taking one triangle from each K_5, we get ϵn^2 edge-disjoint triangles in G, contradicting (3.14). Therefore, G has at most ϵn^2 copies of K_5.

We now delete one edge from each copy of K_5 in G, and lose at most $3\epsilon n^2$ triangles from G. So, we may assume $t(G) \geq \frac{n^2}{8} + 11\epsilon n^2$, and G is $\{\widehat{P}_4, K_5\}$-free.

- **Step 2:** Cleaning K_5^-'s.

  Suppose G contains a K_5^-. Observe that if we have a new vertex $x \neq a_i$ which is adjacent to two endpoints of any edge of this K_5^-, it would create a copy of $\widehat{P}_4$ (see Figure 3-3 (left)). Thus, the only way two K_5^-'s can intersect in an edge is if they share the same five vertices. This would give us a K_5 in G, a contradiction. Therefore, the copies of K_5^- are all edge-disjoint.

  Hence, if G has more than ϵn^2 copies of K_5^-, we again obtain at least ϵn^2 edge-disjoint triangles in G, contradicting (3.14). So G has at most ϵn^2 copies of K_5^-.

  Deleting one edge from each copy of K_5^- in G, we lose at most $3\epsilon n^2$ triangles in the process. After deletion, we still have $t(G) \geq \frac{n^2}{8} + 8\epsilon n^2$, and we can further assume that G is $\{\widehat{P}_4, K_5^-\}$-free.

- **Step 3:** Cleaning K_4's.

Figure 3-3: (left): K_5^-'s are edge-disjoint; (right): K_4's are edge-disjoint.

First, we claim that any two copies of K_4 in G are edge-disjoint. If not, then they can only intersect in one edge, or three edges. If they intersect in one edge, we find a $\widehat{P}_4$, and otherwise we get a K_5^- in G (see Figure 3-3 (right); the intersecting edges are illustrated in bold). Hence, all K_4's in G are edge-disjoint.

Consequently, if there are more than ϵn^2 copies of K_4 in G, taking one triangle from each copy gives us ϵn^2 edge-disjoint triangles, contradicting (3.14) again.

Now, observe that for every K_4 in G with vertices $\{a, b, c, d\}$, either the edge ab or the edge bc has codegree exactly 2. Otherwise, let $x, y \in V(G)$ be such that xab and ybc are triangles in G. If $x = y$, then $xabcd$ is a K_5^-, and otherwise $xadcy$ is a P_4 in the neighborhood of b. Hence, whenever G contains a K_4, we can remove an edge of codegree 2 from it. Then G loses at most $2\epsilon n^2$ triangles. Thus, we assume that $t(G) \geq \frac{n^2}{8} + 6\epsilon n^2$, and that G is $\{\widehat{P}_4, K_4\}$-free.

- **Step 4:** Cleaning $K_{2,2,2}$'s.

  By assumption, G contains no copy of $\widehat{P}_4$ and K_4. Fix a $K_{2,2,2}$ of G with vertices c_1, c_2 in the center and a_1, a_2, a_3, a_4 forming the outer C_4. Let $X = V(G) \setminus \{c_1, c_2, a_1, a_2, a_3, a_4\}$. Denote $C_i = N_G(c_i) \cap X$ for $i = 1, 2$, and $A_i = N_G(a_i) \cap X$ for $i = 1, \ldots, 4$. Since G is $\widehat{P}_4$-free, we deduce that $A_i \cap A_{i+1} = \varnothing$ and $A_i \cap C_j = \varnothing$ for every i, j (here we denote $A_5 := A_1$). This is shown in Figure 3-4, by assuming $x \in A_1 \cap A_2$ and then $x \in A_1 \cap C_2$, and finding copies of $\widehat{P}_4$ in either case. This implies that the $K_{2,2,2}$'s are themselves triangle blocks of G, hence they are mutually edge-disjoint.

  So, if G has at least ϵn^2 copies of $K_{2,2,2}$, then by taking one triangle from each $K_{2,2,2}$ we obtain at least ϵn^2 edge-disjoint triangles in G, contradicting (3.14).

  Deleting one edge from each $K_{2,2,2}$, we lose at most $2\epsilon n^2$ triangles from G. Thus,

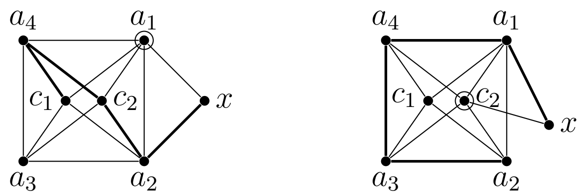

Figure 3-4: $W_{4,2}$ is a block by itself.

we may assume that $t(G) \geq \frac{n^2}{8} + 4\epsilon n^2$, and that G is $\{\widehat{P}_4, K_4, K_{2,2,2}\}$-free.

- **Step 5:** Cleaning $Q_{3,2}$'s.

  Suppose G contains a $Q_{3,2}$ with the P_3 given by vertices a_1, c_1, c_2, a_3 and the outer C_4 being $a_1a_2a_3a_4$. Then, if a_1c_2 or a_3c_1 or a_2a_4 is an edge, we get a K_4 in G, and if a_1a_3 is an edge, then the 4-cycle $a_1c_1c_2a_3$ along with vertices a_2, a_4 create a $K_{2,2,2}$ in G. Hence, every copy of $Q_{3,2}$ in G has to be induced.

  Suppose $X = V(G) \setminus \{a_1, a_2, a_3, a_4, c_1, c_2\}$, and let $C_i = N_G(c_i) \cap X$ for $i = 1, 2$ and $A_i = N_G(a_i) \cap X$ for $i = 1, \ldots, 4$. Since G is $\widehat{P}_4$-free, we deduce that $A_i \cap A_{i+1} = \varnothing$ and $A_i \cap C_j = \varnothing$ for every i, j (here we denote $A_5 := A_1$), and $C_1 \cap C_2 = \varnothing$. We illustrate this in Figure 3-5, similar to before. Hence, the $Q_{3,2}$'s of G are themselves triangle blocks in G.

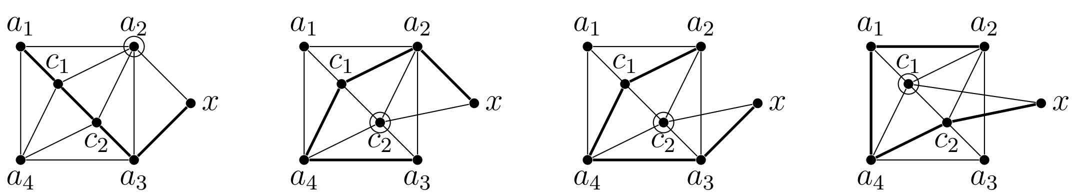

Figure 3-5: $Q_{3,2}$ is a block by itself.

  Consequently, if G has more than ϵn^2 copies of $Q_{3,2}$, then taking one triangle from each $Q_{3,2}$ we obtain at least ϵn^2 edge-disjoint triangles in G, again contradicting (3.14). We delete an outer edge from each copy of $Q_{3,2}$, losing at most ϵn^2 triangles of G. Hence, we can assume that $t(G) \geq \frac{n^2}{8} + 3\epsilon n^2$, and that G is $\{\widehat{P}_4, K_4, K_{2,2,2}, Q_{3,2}\}$-free.

- **Step 6:** Cleaning $K_{1,2,2}$'s.

We proceed similarly as before. First, if G contains a $K_{1,2,2}$ with center c and outer cycle $a_1a_2a_3a_4$, then one cannot have an edge a_1a_3 or a_2a_4 since these give rise to K_4's through c. Hence, the $K_{1,2,2}$'s in G are induced. Plus, none of the edges a_ic lie in a new triangle since it leads to a $\widehat{P}_4$: they all have codegree 2. We now do a case analysis to see that the $K_{1,2,2}$'s in G are edge-disjoint. Let $A, B \in \binom{V(G)}{5}$ be such that $G[A]$ and $G[B]$ are two $K_{1,2,2}$'s which are not edge-disjoint. Then $2 \leq |A \cap B| \leq 4$. Let the central vertices of $G[A]$ and $G[B]$ be u and v, respectively. Since each central edge of $G[A]$ and $G[B]$ has codegree 2, $u \neq v$.

Suppose $|A \cap B| = 2$. If $u \in A \cap B$, then the edge through u with its other endpoint in $A \cap B$ must have codegree at least 3. Thus, the central vertices of $G[A]$ and $G[B]$ must lie outside $A \cap B$, leading us to the first configuration in Figure 3-6. But this configuration admits a P_4 in the neighborhood of either vertex of $A \cap B$, a contradiction. We illustrate $G[A \cap B]$ in boldface.

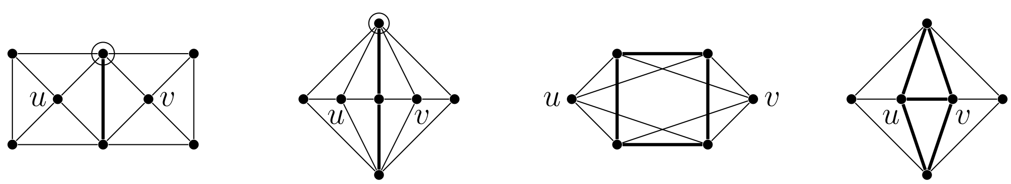

Figure 3-6: The different ways two induced W_4's can intersect.

Next, suppose $|A \cap B| = 3$. If $u \in A \cap B$ but $v \notin A \cap B$, then one of the central edges of $G[A]$ contains an external triangle through v. If $u, v \in A \cap B$, then as u is part of the outer C_4 of $G[B]$, uv contains a triangle from $G[B]$ which is not contained in $G[A]$. Thus the only possibility for $|A \cap B| = 3$ is for u and v to be both outside $A \cap B$. This gives rise to the second configuration in Figure 3-6, which contains $\widehat{P}_4$ in the neighborhood of one of the vertices of $A \cap B$.

Finally, if $|A \cap B| = 4$ and $u \in A \cap B$ but $v \notin A \cap B$, then any edge of $G[A \cap B]$ through u has codegree at least 3. If both u and v lie outside $A \cap B$, we obtain a $K_{2,2,2}$, which is the third configuration in Figure 3-6. Hence, u and v must both lie inside $A \cap B$. Since u and v both must be adjacent to all other vertices of $A \cap B$, $G[A]$ and $G[B]$ form a $Q_{3,2}$, the fourth configuration in Figure 3-6.

Therefore, if two $K_{1,2,2}$'s are not edge-disjoint, they must intersect each other in one of the ways depicted in Figure 3-6, and we either find a $\widehat{P}_4$, $K_{2,2,2}$ or $Q_{3,2}$ inside G for each of these intersecting patterns. Thus, all $K_{1,2,2}$'s of G are edge-disjoint. Consequently, if G has ϵn^2 copies of $K_{1,2,2}$, they are all edge-disjoint, and give us at least ϵn^2 edge-disjoint triangles, again contradicting (3.14).

For each $K_{1,2,2}$ in G with central vertex x and outer cycle $abcd$, we observe that either ab or bc has codegree 1. Otherwise, suppose $y, z \in V(G)$ are such that yab and zbc form triangles in G. If $y = z$, this creates a $Q_{3,2}$ in G. Otherwise, $yaxcz$ is a P_4 in the neighborhood of b. So, every $K_{1,2,2}$ has an outer edge of codegree 1. By deleting one such edge of codegree 1 from each copy of $K_{1,2,2}$, we remove at most ϵn^2 triangles from G. Therefore, we may assume that G is $\{\widehat{P}_4, K_4, K_{1,2,2}\}$-free, and

$$t(G) \geq \frac{n^2}{8} + \epsilon n^2.$$

Let us now analyze the structure of G. We will prove using induction on $t(H)$, that for any subgraph $H \subseteq G$,

$$t(H) \leq \frac{e(H)}{2}. \tag{3.15}$$

When $t(H) = 1$, $e(H) \geq 3$, proving the base case. Now suppose $t(H) > 1$ for some $H \subseteq G$, and that (3.15) holds for all subgraphs H' with $t(H') < t(H)$. Assume without loss of generality that every edge of H lies in at least one triangle. Call an edge of H *light* if it is contained in a unique triangle from H. Call edges that are not light, *heavy*. We observe that if H contains a triangle with two light edges, then deleting them from H leads to a graph $H' \subsetneq H$ with $t(H') = t(H) - 1$ and $e(H') = e(H) - 2$. Using the induction hypothesis on H', $t(H') \leq e(H')/2$, implying $t(H) \leq e(H)/2$. Hence, we may further assume that H contains no triangle with two light edges.

Lemma 3.6.8. *Suppose H contains two triangles xuv and yuv intersecting in the edge uv. Then either: (a) xu, yv are light and xv, yu are heavy or: (b) xu, yv are* *heavy and xv, yu are light.*

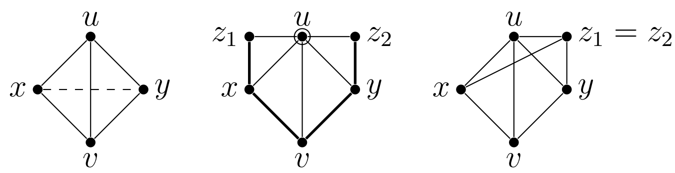

Figure 3-7: xu and yu cannot both be heavy.

Proof of Lemma 3.6.8. Suppose that both xu and yu were heavy (Figure 3-7). If x and y were adjacent, this would create a K_4 which is forbidden. So, there exist $z_1, z_2 \in V(H)$ such that z_1xu and z_2yu form K_3's in H. If $z_1 \neq z_2$, $N(u)$ contains a P_4, which is forbidden. Otherwise $z_1 = z_2$, and this produces a $K_{1,2,2}$ centered at u, a contradiction.

Hence one of xu and yu is light. Similarly, one of xv and yv is light. If xu is light, then xv and yu are heavy, implying that yv is light, and *(a)* holds. Similarly, if xu is heavy, then *(b)* holds.

We shall now use Lemma 3.6.8 and the fact that every triangle in H has two heavy and one light edge, to analyze the structure of H. First, observe that H cannot have any edge of codegree more than 2. This is because if we have an edge uv which lies in three triangles xuv, yuv, zuv, then by Lemma 3.6.8, either xu, yv are light or xv, yu are light. Suppose without loss of generality that xu and yv are light, as in Figure 3-8. Then, by applying Lemma 3.6.8 on the pairs $\{xuv, zuv\}$ and $\{yuv, zuv\}$ respectively, the edges zv and zu must be light. However, this contradicts the assumption of H containing no triangle with two light edges.

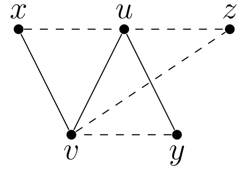

Figure 3-8: Codegree of $uv \in E(H)$ is at most 2.

Now, let $\ell(H)$ denote the number of light edges of H and $h(H)$ the number of heavy edges of H. Since every edge of H can have codegree 1 or 2, and every triangle contains one light and two heavy edges, a double-counting argument gives,

$$\ell(H) + 2h(H) = 3t(H).$$

On the other hand, every light edge of H lies in a unique triangle, and every triangle contains a unique light edge. This implies $t(H) = \ell(H)$. Therefore,

$$4t(H) = 2\ell(H) + 2h(H) = 2e(H),$$

implying $t(H) = \frac{e(H)}{2}$. This finishes the induction step, completing the proof of (3.15).

Taking $H = G$ in (3.15), we obtain $t(G) \leq \frac{e(G)}{2}$. By assumption, $t(G) \geq \frac{n^2}{8} + \epsilon n^2$. So, by Lemma 3.3.2,

$$\frac{e(G)}{2} \geq t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

leading to $e(G) \leq \frac{n^2}{4} + \frac{3n}{8}$. This gives $t(G) \leq \frac{e(G)}{2} \leq \frac{n^2}{8} + \frac{3n}{16}$, which contradicts the assumption of $t(G) \geq \frac{n^2}{8} + \epsilon n^2$ for sufficiently large n. This concludes the proof of Theorem 3.2.4 for $k = 4$. □

Proof of Theorem 3.2.4 for $\mathbf{k = 5}$.

Our proof of Theorem 3.2.4 for $k = 5$ follows exactly the same structure as that for $k = 3$ and $k = 4$, with more technical details. We shall prove, using induction on n, that if G is $\widehat{P}_5$-free, then

$$t(G) \leq \frac{n^2}{4} + 5n. \tag{3.16}$$

The base case $n = 3$ is clearly true as $t(G) \leq 1$. Assume that (3.16) holds for all graphs G on less than n vertices, and let us prove that it also holds for G. The first step is to remove all copies of K_6 and K_6^- from G via the induction hypothesis.

Suppose G has a copy of K_6 with vertices $a_1, \ldots, a_6$. Then this is a block by itself, since if there is a vertex $x \neq a_i$ such that xa_1a_2 is a triangle, then $N_G(a_1)$ contains the 5-path $a_6a_5a_4a_3a_2x$. For $1 \leq i \leq 6$, let $X_i = N_G(a_i) \setminus \{a_1, \ldots, a_6\}$. Then $X_i \cap X_j = \varnothing$ for every $i \neq j$. Since $\sum_{i=1}^{6} |X_i| \leq n - 6$, there is a vertex a_i for which $|X_i| \leq \frac{n-6}{6}$. By Lemma 3.3.3,

$$e(X_i) \leq \frac{5-1}{2} \cdot |X_i| \leq \frac{n-6}{3}.$$

Hence, by (3.16) on $G' = G - \{a_i\}$, we get $t(G') \leq \frac{(n-1)^2}{4} + 5(n-1)$. Therefore,

$$t(G) \leq t(G') + \frac{n-6}{3} + 5 \leq \frac{(n-1)^2}{4} + 5(n-1) + \frac{n-6}{3} + 5 < \frac{n^2}{4} + 5n,$$

completing the induction step for G. Hence, we may assume that G is K_6-free.

Now, if G has a copy of K_6^- on vertices $a_1, \ldots, a_6$, it has to be induced. We verify in Figure 3-9 that it is a block by finding a $\widehat{P}_5$ whenever any edge lies in an external triangle. Let $X_i = N_G(a_i) \setminus \{a_1, \ldots, a_6\}$. Following exactly the same argument as before, there exists a vertex a_i for which $|X_i| \leq \frac{n-6}{6}$. Therefore, applying Lemma 3.3.3 and letting $G' = G - \{a_i\}$,

$$t(G) \leq t(G') + \frac{n-6}{3} + 5 < \frac{n^2}{4} + 5n,$$

completing the induction step for G.

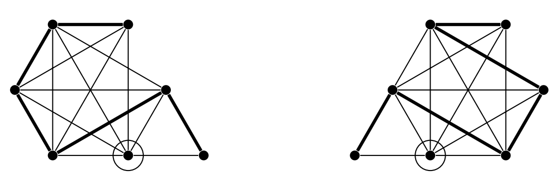

Figure 3-9: K_6^- is a block by itself.

Therefore, without loss of generality we can assume that G is $\{K_6^-, \widehat{P}_5\}$-free. Consider G to be a fixed n-vertex graph. We shall now prove using induction on $t(H)$, that for any subgraph $H \subseteq G$,

$$t(H) \leq e(H). \tag{3.17}$$

When $t(H) = 1$, $e(H) \geq 3$ proves the base case. Now suppose $t(H) > 1$ for some $H \subseteq G$, and that (3.17) holds for all subgraphs H' of G with $t(H') < t(H)$. If H has an edge e which lies in at most one triangle, using the induction hypothesis on $H' = H - \{e\}$ immediately proves (3.17) for H. Hence, we may assume that all edges of H have codegree at least 2. Call an edge of H *light* if it has codegree exactly 2, otherwise call it *heavy*.

Lemma 3.6.9. *We may assume that H does not contain W_5 and $K_{1,2,2}$ as subgraphs.*

This lemma is proved by sequentially removing copies of the graphs illustrated in Figure 3-10 from H, and the proof can be found in Appendix B. We shall now assume that Lemma 3.6.9 is true.

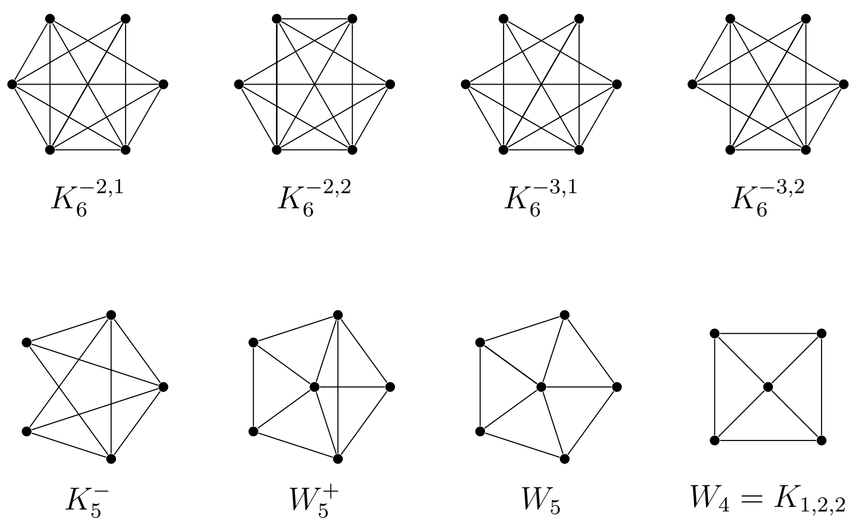

Figure 3-10: Graphs to be cleaned from H.

Suppose H contains a triangle abc such that abx and acy are triangles, with $x \neq y$, i.e H contains a $\widehat{P}_3$. Refer to Figure 3-11. Observe that both ax and ay must have codegree at least 2. If xy is an edge in H, we get a W_4. If axz and ayw are triangles for vertices z and w which are not b or c, then there are two possibilities. Either $z \neq w$ in which case we get a $\widehat{P}_5$, or $z = w$, producing a W_5 in H. Therefore $\{z, w\} \cap \{b, c\} \neq \varnothing$. If $z = c$ and $w = b$, this gives us a $K_{1,2,2}$ centered at a. Hence we may assume $z = c$ and $w \neq b$. By assumption, aw must have codegree at least 2. Note that wb or wx cannot be edges, as they create W_4 or W_5 in H centered around a, respectively.

Further, we cannot have a new vertex t for which awt is a triangle, since this creates a $\widehat{P}_5$ centered at a. Thus, the only possibility is that $wc \in E(H)$.

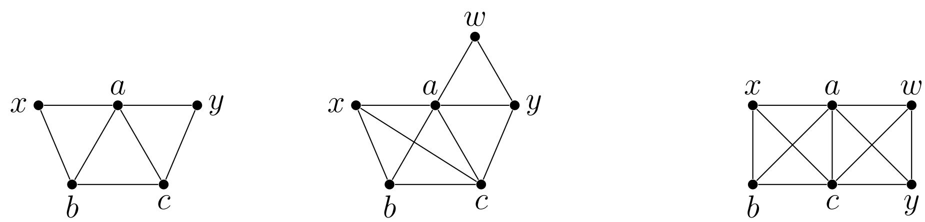

Figure 3-11: abc, abx, acy are triangles with $x \neq y$.

As H is $\{K_{1,2,2}, W_5\}$-free, $H[a, b, c, x, y, w]$ is induced. Further, if the edges ax, ab, ay, aw or cb, cx, cw, cy lie in an external triangle, we can find $\widehat{P}_5$'s centered around a or c, respectively. Hence these 8 edges all do not lie in external triangles, and have codegree exactly 2. Deleting them from H, we obtain a graph H' with $t(H') = t(H) - 8$ and $e(H') = e(H) - 8$, completing the proof of (3.17) for H.

Hence, we may assume that H does not contain any $\widehat{P}_3$. Now if ab and ac were heavy in any triangle abc, we would then find $x \neq y$ for which abx and acy are triangles in H. This leads us to the following crucial observation:

$$\text{Every triangle of } H \text{ has at most one heavy edge.} \tag{3.18}$$

Let us fix a triangle abc in H. Let ab and ac be light. As they must have codegree 2, there is a vertex x for which $xa, xb, xc \in E(H)$, as in Figure 3-12. If the edge xa is light, we can then let $H' = H - \{ab, ax, ac\}$. Note that $t(H') = t(H) - 3$ and $e(H') = e(H) - 3$, finishing the proof of (3.17) for H. Finally, if xa is heavy, then by (3.18), the edges xb and xc must be light. Let $H' = H - \{ab, ac, xb, xc\}$, then $t(H') = t(H) - 4$ and $e(H') = e(H) - 4$, completing the induction step of (3.17).

Taking $H = G$ in (3.17), we obtain $t(G) \leq e(G)$. Using Lemma 3.3.2,

$$e(G) \geq t(G) \geq \frac{e(G)}{3n} \cdot (4e(G) - n^2),$$

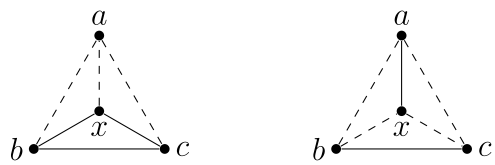

Figure 3-12: Edge xa can be light or heavy.

implying $t(G) \le e(G) \le \frac{n^2}{4} + \frac{3n}{4} < \frac{n^2}{4} + 5n$. This concludes the proof of (3.17) for G.

Chapter 4

EXTREMAL NUMBERS OF HYPERGRAPH SUSPENSIONS OF EVEN CYCLES

4.1 Background

Recall that an r-uniform hypergraph, or simply, an r-graph H on vertex set $V(H)$ is a subset of $\binom{V(H)}{r}$. We consider the Turán problem for 3-graphs in this chapter.

Recall that for $k \geq 2$, $\mathrm{ex}(n, C_{2k}) \leq O(n^{1+\frac{1}{k}})$ [10, 46, 62, 12]. The even cycle problem entails constructing C_{2k}-free graphs on n vertices and $\Omega(n^{1+\frac{1}{k}})$ edges. Tight bounds are only known for $k \in \{2, 3, 5\}$ [63, 11, 5, 52, 72, 49], and for $k \notin \{2, 3, 5\}$, the best known lower bounds are given by the bipartite graphs $CD(k, q)$ for integers $k \geq 2$ and prime powers q [47, 48, 73].

4.2 Our Results

In this chapter, we are mainly concerned with three classes of lower bound constructions for the even cycle problem: the bipartite graphs $D(k, q)$ from [47, 48], the arc construction introduced in [53] and later generalized in [54], and Wenger's construction [72]. Our results can be divided into two sections: results about 3-graphs and results about graphs.

4.2.1 3-Graphs

In Chapter 3, we studied the suspension $\widehat{G}$ of a graph G which is obtained by adding a new vertex adjacent to all vertices of G. Analogously, we introduce the concept of a *hypergraph suspension*.

Let H be a 3-graph and $x \in V(H)$ be any vertex of H. The link of x in H, denoted by $L_{x,H}$, is the graph with vertex set $V(H) \setminus \{x\}$ and edges $\{uv : \{x, u, v\} \in H\}$. For a graph G, the hypergraph suspension $\widetilde{G}$ is a 3-graph defined as follows: add a new vertex x to $V(G)$, and let $\widetilde{G} = \{e \cup \{x\} : e \in E(H)\}$. By definition, $L_{x,\widetilde{G}} = G$.

Note that the numbers $\mathrm{ex}_3(n, \widetilde{G})$ and $\mathrm{ex}(n, K_3, \widehat{G})$ are closely related. In fact, given a $\widehat{G}$-free graph, we can replace all triangles in it with hyperedges to obtain a $\widetilde{G}$-free 3-graph, implying

$$\mathrm{ex}(n, K_3, \widehat{G}) \leq \mathrm{ex}_3(n, \widetilde{G}). \tag{4.1}$$

We focus our attention on $\mathrm{ex}_3(n, \widetilde{C}_{2k})$ for $k \geq 2$. When $k = 2$, observe that $\widetilde{C}_{2k}$ is the complete 3-partite 3-graph $K^{(3)}_{1,2,2}$, and its extremal number has been exactly determined to be $\Theta(n^{5/2})$ in [55]. Thus, we consider $\widetilde{C}_{2k}$ for $k \geq 3$.

Observe that a 3-graph H does not contain $\widetilde{C}_{2k}$ iff $L_{x,H}$ does not contain C_{2k} for every vertex $x \in V(H)$, leading us to the upper bound

$$\mathrm{ex}_3(n, \widetilde{C}_{2k}) \leq O(n \cdot n^{1+\frac{1}{k}}) = O(n^{2+\frac{1}{k}}) \tag{4.2}$$

On the other hand, a probabilistic deletion argument lets us deduce the following result:

Proposition 4.2.1. *For $k \geq 2$,*

$$ex_3(n, \widetilde{C}_{2k}) \geq \Omega\left(n^{2+\frac{1}{2k-1}}\right). \tag{4.3}$$

Our main result is to show a construction of $\widetilde{C}_{2k}-$free 3-graphs, which asymptotically improves the bound above for $k = 3$ and $k = 4$.

Theorem 4.2.2. *For every integer q that is a power of 3, there exists a 3-partite 3-graph $D_3(k, q)$ with the following properties:*

1. *$D_3(k, q)$ has $3q^k$ vertices and q^{2k+1} edges,*
2. *The link graph of every vertex of $D_3(k, q)$ is isomorphic for $k \leq 6$, and*
3. *$D_3(3, q)$ and $D_3(5, q)$ are $\widetilde{C}_6$ and $\widetilde{C}_8$-free, respectively.*

In particular, Theorem 4.2.2 implies that

$$\mathrm{ex}_3(n, \widetilde{C}_6) \geq \Omega(n^{7/3}) \text{ and } \mathrm{ex}_3(n, \widetilde{C}_8) \geq \Omega(n^{11/5}). \tag{4.4}$$

As a corollary of (4.2) and (4.4), we determine the exact growth rate of $\mathrm{ex}_3(n, \widetilde{C}_6)$.

Corollary 4.2.3. For large n, the Turán number of $\widetilde{C}_6$ grows as,

$$\mathrm{ex}_3(n, \widetilde{C}_6) = \Theta(n^{7/3}). \tag{4.5}$$

Corollary 4.2.3 further implies that the bound in (4.1) is not always sharp, since we demonstrated in Theorem 3.2.2 that $\mathrm{ex}(n, K_3, \widehat{C}_6) = o(n^{7/3})$.

Remark. Our proof of Theorem 4.2.2 heavily relies on the bipartite graphs $D(k, q)$ introduced by Lazebnik, Ustimenko and Woldar in [47], and $D_3(k, q)$ can be viewed as an extension of $D(k,q)$ to 3-graphs. $D_3(k,q)$ has the property that for every $k \geq 2$ and prime power $3 \mid q$, the link graph of any of its vertex is isomorphic to either $D(k,q)$ or another graph which we call $D'(k,q)$ (Proposition 4.3.5). We also make a conjecture (Conjecture 4.3.9) about the girth of $D'(k,q)$, which, if true, would give a bound of $\mathrm{ex}_3(n, \widetilde{C}_{2k}) \geq \Omega(n^{2+\frac{1}{2k-3}})$ for all $k \geq 3$, an asymptotic improvement on (4.3).

4.2.2 Graphs

In the second part of this chapter, we compare two well-known constructions of C_{2k}-free graphs: the arc construction [53, 54] and Wenger's construction [72]. Let $t \geq 2$, and let q be a prime power. An arc in a projective t-space $PG(t,q)$ is a collection of points such that no $(t-1)$ of them lie in a hyperplane. The arc construction is defined as follows.

The bipartite graphs $G_{\texttt{arc}}(k,q,\alpha)$. Let $\Sigma = PG(t,q)$, and $\Sigma_0 \subset \Sigma$ be the hyperplane consisting of points with first homogeneous coordinate 0. Note that $\Sigma_0 \cong PG(t-1,q)$. Let α be any arc in Σ_0. Then, the bipartite graph $G_{\texttt{arc}}(k,q,\alpha)$ with parts P and L is defined as follows. Let $P = \Sigma \setminus \Sigma_0$, and L be the set of all projective lines ℓ of Σ such that $\ell \cap \Sigma_0 \in \alpha$. Vertices $p \in P$ and $\ell \in L$ are adjacent if and only if $p \in \ell$.

It was shown in [54] that $G_{\texttt{arc}}(k,q,\alpha_0)$ is C_{2k}-free for $k = 2,3,5$ but contains C_8 for $k = 4$, where α_0 is the normal rational curve in Σ_0 given by

$$\alpha_0 = \{[0:1:x:x^2:\cdots:x^{t-1}] : x \in \mathbb{F}_q\} \cup \{[0:0:\cdots:0:1]\}.$$

In contrast, let $H(k,q)$ be the bipartite graph with parts $A = B = \mathbb{F}_q^k$ such that $(a_1,\ldots,a_k)$ is adjacent to $(b_1,\ldots,b_k)$ iff

$$a_i + b_i = a_1 b_1^{i-1} \text{ for all } 2 \leq i \leq k.$$

It was shown by Wenger in [72] that $H(k,q)$ is C_{2k}-free for $k = 2, 3, 5$.

We prove that these two constructions are in fact, isomorphic, and our proof uses the Plücker embedding [37], a tool from algebraic geometry that lets us parametrize the set of projective lines L.

Proposition 4.2.4. *Let α_0 be the normal rational curve in $PG(2,k)$, and $\alpha_0^- = \alpha_0 \setminus \{[0 : \cdots : 0 : 1]\}$. Then,*

$$G_{\mathtt{arc}}(k, q, \alpha_0^-) \cong H(k, q).$$

As $G_{\mathtt{arc}}(4, q, \alpha_0)$ is shown to contain C_8's in [54], Proposition 4.2.4 also provides a geometric explanation for why Wenger's bound is tight for $k = 3$ and $k = 5$ but not $k = 4$.

For $1 \le s \le r$ with $(s,r) = 1$, it is known that $\alpha = \{[1 : x : x^{2^s}] : x \in \mathbb{F}_{2^r}\}$ is an arc in the projective space $PG(2, 2^r)$. Using the proof method of Proposition 4.2.4 on this arc α, we are able to construct a new family of C_6-free graphs with $\Omega(n^{4/3})$ edges, given as follows.

Theorem 4.2.5. *Let $q = 2^r$ and $1 \le s \le r$ be such that $(s,r) = 1$. Let $G(2^r, s)$ denote the bipartite graph with parts $A = B = \mathbb{F}_q^3$ such that $(a_1, a_2, a_3) \in A$ is adjacent to $(b_1, b_2, b_3) \in B$ iff*

$$b_2 + a_2 = a_1 b_1 \text{ and } b_3 + a_3 = a_1 b_1^{2^s}.$$

Then, $G(2^r, s)$ is C_6-free.

Note that the graphs $G(2^r, s)$ extend Wenger's C_6-free construction in even characteristic, as $G(2^r, 1) \cong H(3, 2^r)$.

This chapter is organized as follows. In Section 2, we prove Proposition 4.2.1, recapitulate on the graphs $D(k,q)$, extend them to the 3-graphs $D_3(k,q)$, and investigate its link graphs, finally proving Theorem 4.2.2. Section 3 is devoted to proving Proposition 4.2.4 and Theorem 4.2.5.

4.3 Lower bounds on $\mathbf{ex}(n, \widetilde{C}_{2k})$

Our goal in this section is to extend the graphs $D(k,q)$ to a family of 3-graphs, and build up the tools required to prove Theorem 4.2.2. We start with a proof of Proposition 4.2.1. Recall that we wish to show $\mathrm{ex}_3(n, \widetilde{C}_{2k}) \geq \Omega(n^{2+\frac{1}{2k-1}})$.

Proof of Proposition 4.2.1. Let $H \sim G_3(n,p)$ be the Erdős-Rényi 3-graph, where each edge of the complete 3-graph on n vertices is selected with probability $p = \frac{1}{10}k^{-100}n^{-\frac{2k-2}{2k-1}}$. Then, $\mathbb{E}(|H|) = p\binom{n}{3}$. For every $\widetilde{C}_{2k}$ in H, we remove one edge from it. Let $H' \subset H$ be the new 3-graph obtained via the deletion of edges. Note that the probability that any $2k+1$ vertices forms a $\widetilde{C}_{2k}$ is $(2k+1)p^{2k}$, and therefore, the expected number of them is at most $(2k+1)n^{2k+1}p^{2k}$. Now, $E(H') = p\binom{n}{3} - (2k+1)n^{2k+1}p^{2k}$. As

$$n^{2k+1}p^{2k-1} = n^{2k+1} \cdot 10^{-(2k-1)}k^{-100(2k-1)}n^{-(2k-2)} \leq 10^{-(2k-1)}n^3k^{-100(2k-1)},$$

and

$$p\left(\binom{n}{3} - (2k+1)n^{2k+1}p^{2k}\right) \geq pn^3\left(\frac{1}{10} - \frac{2k+1}{10^{2k-1}}\right) \geq \frac{pn^3}{100},$$

This implies that $\mathbb{E}(|H'|) \geq \frac{1}{1000}k^{-100}n^{3-\frac{2k-2}{2k-1}}$. Thus, there exists a 3-graph H' with $\Omega(n^{3-\frac{2k-2}{2k-1}})$ edges with no copy of $\widetilde{C}_{2k}$. This completes our proof. □

Since probabilistic lower bounds for 3-graphs tend to be weak, we try to strengthen this result via the graphs $D(k,q)$. Here we present a summary of the properties of $D(k,q)$; for more details, the reader is referred to [47, 48, 73].

Definition 4.3.1 (The bipartite graphs $D(q)$)**.** For a prime power q, let A and B be two copies of the countably infinite dimensional vector space V over $\mathbb{F}_q$. Use the following coordinate representations for elements $a \in A$ and $b \in B$:

$$\begin{aligned} a &= (a_1, a_{11}, a_{12}, a_{21}, a_{22}, a'_{22}, a_{23}, \ldots, a_{ii}, a'_{ii}, a_{i,i+1}, a_{i+1,i}, \ldots), \\ b &= (b_1, b_{11}, b_{12}, b_{21}, b_{22}, b'_{22}, b_{23}, \ldots, b_{ii}, b'_{ii}, b_{i,i+1}, b_{i+1,i}, \ldots). \end{aligned} \tag{4.6}$$

Let $A \sqcup B$ be the vertex set of $D(q)$, and join $a \in A$ to $b \in B$ if the following coordinate relations hold ($i \geq 2$):

$$
\begin{aligned}
a_{11} + b_{11} + a_1 b_1 &= 0 \\
a_{12} + b_{12} + a_1 b_{11} &= 0 \\
a_{21} + b_{21} + a_{11} b_1 &= 0 \\
&\vdots \\
a_{ii} + b_{ii} + a_{i-1,i} b_1 &= 0 \\
a'_{ii} + b'_{ii} + a_1 b_{i,i-1} &= 0 \\
a_{i,i+1} + b_{i,i+1} + a_1 b_{ii} &= 0 \\
a_{i+1,i} + b_{i+1,i} + a'_{ii} b_1 &= 0.
\end{aligned}
\tag{4.7}
$$

Define $D(k, q)$ to be the graph obtained by truncation of A and B to the first k coordinates in (4.6) and, the first $k - 1$ relations in (4.7).

The key properties of the graphs $D(k, q)$ are summarized in the following proposition.

Proposition 4.3.2. *For any prime power q and $k \geq 2$, the following holds:*

1. *$D(k, q)$ is a q-regular bipartite graph of order $2q^k$;*

2. *The girth of $D(k, q)$ is at least $k + 4$ if k is even, and $k + 5$ if k is odd.*

Further, it is known that for $k \geq 6$ the graphs $D(k, q)$ start to get disconnected into pairwise isomorphic components at regular intervals. These connected components are called $CD(k, q)$. The graphs $CD(2k - 3, q)$ give the currently best known asymptotic lower bounds on $\mathrm{ex}(n, C_{2k})$ for $k \geq 3$. We omit the proof of Proposition 4.3.2 here.

In the following subsection, we extend $D(k, q)$ to the 3-graph case.

4.3.1 The 3-graphs $D_3(k,q)$

Definition 4.3.3 (The 3-partite 3-graphs $D_3(q)$)**.** For a prime power q, let A, B, and C be three copies of the countably infinite dimensional vector space V over $\mathbb{F}_q$. We use the following coordinate representations for $a \in A$, $b \in B$, $c \in C$:

$$a = (a_1, a_{11}, a_{12}, a_{21}, a_{22}, a'_{22}, a_{23}, \ldots, a_{ii}, a'_{ii}, a_{i,i+1}, a_{i+1,i}, \ldots),$$
$$b = (b_1, b_{11}, b_{12}, b_{21}, b_{22}, b'_{22}, b_{23}, \ldots, b_{ii}, b'_{ii}, b_{i,i+1}, b_{i+1,i}, \ldots),$$
$$c = (c_1, c_{11}, c_{12}, c_{21}, c_{22}, c'_{22}, c_{23}, \ldots, c_{ii}, c'_{ii}, c_{i,i+1}, c_{i+1,i}, \ldots).$$

Let $A \sqcup B \sqcup C$ be the vertex set of $D(q)$, and say that $\{a,b,c\}$ is a hyperedge if the following coordinate relations (call them I) hold ($i \geq 2$):

$$\begin{aligned}
a_{11} + b_{11} + c_{11} + a_1 b_1 + b_1 c_1 + c_1 a_1 &= 0\\
a_{12} + b_{12} + c_{12} + a_1 b_{11} + b_1 c_{11} + c_1 a_{11} &= 0\\
a_{21} + b_{21} + c_{21} + a_{11} b_1 + b_{11} c_1 + c_{11} a_1 &= 0\\
&\vdots
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
a_{ii} + b_{ii} + c_{ii} + a_{i-1,i} b_1 + b_{i-1,i} c_1 + c_{i-1,i} a_1 &= 0\\
a'_{ii} + b'_{ii} + c'_{ii} + a_1 b_{i,i-1} + b_1 c_{i,i-1} + c_1 a_{i,i-1} &= 0\\
a_{i,i+1} + b_{i,i+1} + c_{i,i+1} + a_1 b_{ii} + b_1 c_{ii} + c_1 a_{ii} &= 0\\
a_{i+1,i} + b_{i+1,i} + c_{i+1,i} + a'_{ii} b_1 + b'_{ii} c_1 + c'_{ii} a_1 &= 0.
\end{aligned}$$

Define $D_3(k,q)$ to be the 3-graph obtained by truncation of A, B, and C to the first k coordinates and I to the first $k-1$ relations.

The graphs $D_3(q)$ are designed in such a way that the link of the vertex $\vec{0}$ from any part is isomorphic to $D(q)$. In fact, note that $D_3(q)$ has the natural cyclic automorphism $a_* \mapsto b_*$, $b_* \mapsto c_*$, and $c_* \mapsto a_*$, under which all the defining equations of $D_3(q)$ remain invariant. Hence, for any vertex $v \in V$, the links of v in $D_3(q)$ taken from either of the three parts are all isomorphic, and it is enough to consider the link graphs from a fixed part, say, A.

One would hope that the link graphs of other vertices in $D_3(k,q)$ also have similar high girth properties as $D(k,q)$. This inspires us to analyze the links of every vertex in $D_3(k,q)$. To that end, we analyze $\mathrm{Aut}(D_3(q))$.

Proposition 4.3.4. *Suppose $\mathbb{F}_q$ has characteristic 3, and consider $D_3(q)$ with parts A, B, C. Let $a \in A$ be fixed, and suppose $s \geq 1$. Then there is an automorphism $\varphi \in Aut(D_3(q))$ such that $\varphi(a) = (a_1, 0, \ldots, 0, *, *, \ldots) \in A$, where the second through the $(s+1)$'th coordinates are mapped to 0.*

The proof of Proposition 4.3.4 is technical. Before looking at the proof, we note an important consequence: it is sufficient to analyze the girths of the link graphs of the vertices $(a_1, 0, \ldots, 0) \in A$ for $a_1 \in \mathbb{F}_q$. In fact, it is seen that the truncated 3-graphs $D_3(k,q)$ have exactly two kinds of links.

Proposition 4.3.5. *If $3 \mid q$, then the 3-graph $D_3(k,q)$ admits exactly two classes of link graphs, one of which is $D(k,q)$.*

Now, we present the proofs of Propositions 4.3.4 and 4.3.5.

Proof of Proposition 4.3.4

Recall that $3 \mid q$, and we wish to construct an automorphism φ of $D(q)$ sending any vertex $a \in A$ to $(a_1, 0, \ldots, 0, *, *, \ldots) \in A$, where there are s zeros followed by a_1.

We construct φ via a product of automorphisms of $D_3(q)$. First, we may rewrite the system of equations (4.8) into the following form:

$$\left.\begin{aligned} a_{ii} + b_{ii} + c_{ii} + a_{i-1,i}b_1 + b_{i-1,i}c_1 + c_{i-1,i}a_1 = 0 \\ a'_{ii} + b'_{ii} + c'_{ii} + a_1 b_{i,i-1} + b_1 c_{i,i-1} + c_1 a_{i,i-1} = 0 \\ a_{i,i+1} + b_{i,i+1} + c_{i,i+1} + a_1 b_{ii} + b_1 c_{ii} + c_1 a_{ii} = 0 \\ a_{i+1,i} + b_{i+1,i} + c_{i+1,i} + a'_{ii} b_1 + b'_{ii} c_1 + c'_{ii} a_1 = 0 \end{aligned}\right\} , i \geq 1, \tag{4.9}$$

where we set the convention $a_{01} = a_1, b_{01} = b_1, c_{01} = c_1$; $a'_{11} = a_{11}, b'_{11} = b_{11}, c'_{11} = c_{11}$; and $a_{10} = a_1, b_{10} = b_1, c_{10} = c_1$, with the implication that the first and second equations coincide for $i = 1$. Further, for the sake of ease in defining the automorphisms, we give meaningful interpretations for the equations in (4.9) when $i = 0$. We set $a'_{00} = b'_{00} = c'_{00} = a_{00} = b_{00} = c_{00} = -1$; and $a_{0,-1} = b_{0,-1} = c_{0,-1} = a_{-1,0} = b_{-1,0} = c_{-1,0} = 0$. Notice that the first and the second equations reduce to $-3 = 0$ for $i = 0$, which is true in characteristic 3.

Coordinates ($i \geq 0$)	$t_{1,1}(x)$	$t_{m,m+1}(x), m \geq 1$ $r = i - m$	$t_{m+1,m}(x), m \geq 1$ $r = i - m$	$t_{m,m}(x), m \geq 2$ $r = i - m$	$t'_{m,m}(x), m \geq 2$ $r = i - m$
a_{ii}	$+a_{i-1,i-1}x$	$+a_{r,r-1}x$, $r \geq 1$	-	$+a_{rr}x$, $r \geq 0$	-
$a_{i,i+1}$	$+a_{i-1,i}x$	$+a'_{rr}x$, $r \geq 0$	-	$+a_{r,r+1}x$, $r \geq 0$	-
$a_{i+1,i}$	$+a_{i,i-1}x$	-	$+a_{rr}x$, $r \geq 0$	-	$+a_{r+1,r}x$, $r \geq 0$
a'_{ii}	$+a'_{i-1,i-1}x$	-	$+a_{r-1,r}x$, $r \geq 1$	-	$+a'_{rr}x$, $r \geq 0$
b_{ii}	$+b_{i-1,i-1}x$	$+b_{r,r-1}x$, $r \geq 1$	-	$+b_{rr}x$, $r \geq 0$	-
$b_{i,i+1}$	$+b_{i,i-1}x$	$+b'_{rr}x$, $r \geq 0$	-	$+b_{r,r+1}x$, $r \geq 0$	-
$b_{i+1,i}$	$+b_{i,i-1}x$	-	$+b_{rr}x$, $r \geq 0$	-	$+b_{r+1,r}x$, $r \geq 0$
b'_{ii}	$+b'_{i-1,i-1}x$	-	$+b_{r-1,r}x$, $r \geq 1$	-	$+b'_{rr}x$, $r \geq 0$
c_{ii}	$+c_{i-1,i-1}x$	$+c_{r,r-1}x$, $r \geq 1$	-	$+c_{rr}x$, $r \geq 0$	-
$c_{i,i+1}$	$+c_{i-1,i}x$	$+c'_{rr}x$, $r \geq 0$	-	$+c_{r,r+1}x$, $r \geq 0$	-
$c_{i+1,i}$	$+c_{i,i-1}x$	-	$+c_{rr}x$, $r \geq 0$	-	$+c_{r+1,r}x$, $r \geq 0$
c'_{ii}	$+c'_{i-1,i-1}x$	-	$+c_{r-1,r}x$, $r \geq 1$	-	$+c'_{rr}x$, $r \geq 0$

Table 4.1: Automorphisms of $D(q)$

($a'_{00} = b'_{00} = c'_{00} = a_{00} = b_{00} = c_{00} = -1$, $a_{0,-1} = b_{0,-1} = c_{0,-1} = a_{-1,0} = b_{-1,0} = c_{-1,0} = 0$)

Now, we define five different automorphisms of $D(q)$ in Table 4.1 below, by noting where each coordinate is sent to. For example, for fixed $x \in \mathbb{F}_q$, we denote $t_{1,1}(x)$ to be the automorphism that sends $a_1 \mapsto a_1 + a_{-1,0}x = a_1$, $a_{11} \mapsto a_{11} + a_{00}x = a_{11} - x$, and so on. A "-" as a table entry denotes a coordinate fixed by that map, e.g $t_{m+1,m}(a_{ii}) = a_{ii}$.

Claim 4.3.6. The maps defined in Table 4.1 are Automorphisms of $D(q)$.

Proof of Claim 4.3.6. Observe that each of the maps defined have inverses given by x replaced with $-x$, respectively, once we check that they are homomorphisms.

- $t_{1,1}(x)$: We observe that the map $t_{1,1}(x)$ keeps a_1, b_1, c_1 fixed as $a_1 = a_{0,1} \mapsto a_{0,1} + a_{-1,0}x = a_{0,1}$, etc. And, for $i \geq 1$, we need to check that the equations (4.9) are preserved after the transformation given by $t_{1,1}$. Suppose the equations (4.9) hold, then note that we also have for $i \geq 1$,

$$a_{ii} + b_{ii} + c_{ii} + a_{i-1,i}b_1 + b_{i-1,i}c_1 + c_{i-1,i}a_1 = 0,$$
$$(a_{i-1,i-1} + b_{i-1,i-1} + c_{i-1,i-1} + a_{i-2,i-1}b_1 + b_{i-2,i-1}c_1 + c_{i-2,i-1}a_1)x = 0,$$

  and adding these up verifies that the first equation is preserved under the image of $t_{1,1}(x)$. Similarly, the other three equations can be verified for each $i \geq 1$.

- $t_{m,m+1}(x), m \geq 1$: Again, note that this map fixes $a_1 = a_{0,1}$, $b_1 = b_{0,1}$ and $c_1 = c_{0,1}$ as for $i = 0$ and $m \geq 1$, $r = i - m < 0$. It also fixes all a_{ii}, $i \leq m$ and all $a_{i,i+1}$, $i < m$. Therefore, all of (4.9) are satisfied for $i < m$. When $i = m$, the first equation is still preserved as $a_{mm}, a'_{m-1,m}$ are fixed. For the third equation, we observe that $a_{m,m+1} \mapsto a_{m,m+1} + a'_{00}x = a_{m,m+1} - x$, $b_{m,m+1} \mapsto b_{m,m+1} - x$ and $c_{m,m+1} \mapsto c_{m,m+1} - x$. Thus, the third equation becomes

$$(a_{m,m+1} - x) + (b_{m,m+1} - x) + (c_{m,m+1} - x) + a_1 b_{mm} + b_1 c_{mm} + c_1 a_{mm} = 0,$$

  which is still true as $3x = 0$ in $\mathbb{F}_q$. Finally, for $i > m$, we need to check the validity of the first and third equations from (4.9). However, note that for $i > m$

and $r = i - m \geq 1$,

$$a_{ii} + b_{ii} + c_{ii} + a_{i-1,i}b_1 + b_{i-1,i}c_1 + c_{i-1,i}a_1 = 0,$$
$$(a_{r,r-1} + b_{r,r-1} + c_{r,r-1} + a'_{r-1,r-1}b_1 + b'_{r-1,r-1}c_1 + c'_{r-1,r-1}a_1)x = 0,$$

and adding these up verifies the first equation, since $t_{m,m+1}(x)(a_{i-1,i}) = a_{i-1,i} + a'_{r-1,r-1}x$. In a similar fashion, we verify the third equation by adding up:

$$a_{i,i+1} + b_{i,i+1} + c_{i,i+1} + a_1b_{ii} + b_1c_{ii} + c_1a_{ii} = 0,$$
$$(a'_{rr} + b'_{rr} + c'_{rr} + a_1b_{r,r-1} + b_1c_{r,r-1} + c_1a_{r,r-1})x = 0,$$

for $i > m$ and $r = i - m \geq 1$. The second and fourth equations are unchanged by $t_{m,m+1}$.

- $t_{m+1,m}(x), m \geq 1$: Similar to $t_{m,m+1}$, this map fixes a_{ii} and $a_{i,i+1}$ for every i, and hence does not change the first and third set of equations of (4.9). It changes $a_{m+1,m} \mapsto a_{m+1,m} - x$, yet fixes a'_{mm}, hence satisfies

$$(a_{m+1,m} - x) + (b_{m+1,m} - x) + (c_{m+1,m} - x) + a'_{mm}b_1 + b'_{mm}c_1 + c'_{mm}a_1 = 0.$$

Finally, when $i > m$, the following four equations vouch for the validity of the second and fourth equations of (4.9):

$$\left\| \begin{array}{c} \left\| \begin{array}{c} a_{i+1,i} + b_{i+1,i} + c_{i+1,i} + a'_{ii}b_1 + b'_{ii}c_1 + c'_{ii}a_1 = 0 \\ (a_{rr} + b_{rr} + c_{rr} + a_{r-1,r}b_1 + b_{r-1,r}c_1 + c_{r-1,r}a_1)x = 0 \end{array} \right\| \\ a'_{ii} + b'_{ii} + c'_{ii} + a_1b_{i,i-1} + b_1c_{i,i-1} + c_1a_{i,i-1} = 0 \\ (a_{r-1,r} + b_{r-1,r} + c_{r-1,r} + a_1b_{r-1,r-1} + b_1c_{r-1,r} + c_1a_{r-1,r})x = 0 \end{array} \right\|.$$

- $t_{m,m}(x), m \geq 2$: Same as before, we start by observing that $t_{m,m}(a_{mm}) = a_{mm} - x$, $t_{m,m}(a_{m-1,m}) = a_{m-1,m}$, preserving the first equation of (4.9) for $i = m$. On the other hand, as $a_{m,m+1} \mapsto a_{m,m+1} + a_{0,1}x = a_{m,m+1} + a_1x$, we can rewrite the third equation into:

$$(a_{m,m+1} + a_1 x) + (b_{m,m+1} + b_1 x) + (c_{m,m+1} + c_1 x) + a_1(b_{mm} - x) \\ + b_1(c_{mm} - x) + c_1(a_{mm} - x) = 0.$$

For $i > m$ and $r = i - m \geq 1$, we only add the first and third equations to themselves for $i = i$ and $i = r$.

- $t'_{m,m}(x), m \geq 2$: For this map, $t'_{m,m}(a'_{mm}) = a'_{mm} - x$, $t'_{m,m}(a_{m,m-1}) = a_{m,m-1}$, verifying the second equation of (4.9) for $i = m$. And, as $t'_{m,m}(a_{m+1,m}) = a_{m+1,m} + a_{1,0}x = a_{m+1,m} + a_1 x$, we again have

$$(a_{m+1,m} + a_1 x) + (b_{m+1,m} + b_1 x) + (c_{m+1,m} + c_1 x) + (a'_{mm} - x)b_1 \\ + (b'_{mm} - x)c_1 + (c'_{mm} - x)a_1 = 0.$$

  For $i > m$ and $r = i - m \geq 1$, adding the first and third equations to themselves for $i = i$ and $i = r$ completes the verification.

This calculation shows that the maps defined in Table 4.1 are all homomorphisms. Since replacing x by $-x$ doesn't change the verification of the equations, and since $f(x) \circ f(-x)$ is the identity map for $f = t_{1,1},\ t_{m,m+1},\ t_{m+1,m},\ t_{m,m}$ and $t'_{m,m}$, this implies that all these maps are automorphisms. This finishes the proof of Claim 4.3.6. ■

We now return to the proof of Proposition 4.3.4. In the proof of Claim 4.3.6, we checked that $t_{1,1}(x)$ keeps a_1 fixed, and moves $a_{11} \mapsto a_{11} + a_{00}x = a_{11} - x$. Therefore, given an edge $\{a, b, c\}$ of $D(q)$, we can perform $t_{1,1}(a_{11})$ to map a_{11} to 0. Let $a^{(11)} = t_{1,1}(a_{11})(a)$. Now, an application of $t_{1,2}(a_{12}^{(11)})$ sends the third coordinate, $a_{12}^{(11)}$ to 0. Let $a^{(12)} = t_{1,2}(a_{12}^{(11)})(a^{(11)})$, and $a^{(21)}$, $a^{(22')}$, $a^{(32)}, \ldots$ be defined similarly. Then, the map φ given by

$$\varphi = \cdots \circ t_{i+1,i}(a_{i+1,i}^{(i+1,i)}) \circ t_{i,i+1}(a_{i+1,i}^{(ii')}) \circ t'_{ii}(a_{ii}^{(ii)\prime}) \circ t_{ii}(a_{ii}^{(i-1,i)}) \circ \cdots \circ t_{1,2}(a_{12}^{(11)}) \circ t_{1,1}(a_{11}),$$

where φ is truncated to s compositions, sends the second through $(s+1)$'th coordinates of a to 0. It also preserves all edges through a, being an automorphism of $D(q)$. This completes the proof. □

Proof of Proposition 4.3.5

Our goal in this section is to prove that $D_3(k, q)$ admits two different link graphs. We shall consider the link graphs of $a = (a_1, 0, \ldots, 0) \in A$ for $a_1 \in \mathbb{F}_q$. Let L_a denote the link graph of a. We see that $bc \in E(L_a)$ if and only if the following equations hold $(i \geq 2)$:

$$\begin{aligned} b_{11} + c_{11} + a_1 b_1 + b_1 c_1 + c_1 a_1 &= 0 \\ b_{12} + c_{12} + a_1 b_{11} + b_1 c_{11} &= 0 \\ b_{21} + c_{21} + b_{11} c_1 + c_{11} a_1 &= 0 \\ &\vdots \\ b_{ii} + c_{ii} + b_{i-1,i} c_1 + c_{i-1,i} a_1 &= 0 \\ b'_{ii} + c'_{ii} + a_1 b_{i,i-1} + b_1 c_{i,i-1} &= 0 \\ b_{i,i+1} + c_{i,i+1} + a_1 b_{ii} + b_1 c_{ii} &= 0 \\ b_{i+1,i} + c_{i+1,i} + b'_{ii} c_1 + c'_{ii} a_1 &= 0. \end{aligned} \tag{4.10}$$

Here we consider two different cases.

- **Case 1:** $a_1 = 0$. In this case, we note that the relations (4.10) reduce to the relations (4.7) defining $D(k, q)$, implying $L_a \cong D(k, q)$.
- **Case 2:** $a_1 \neq 0$. In this case, let us define another automorphism ψ on L_a as follows:

$$\left\{ \begin{array}{rl} \psi(b_1) &= \frac{b_1}{a_1} \\ \psi(b_{ii}) &= \frac{b_{ii}}{a_1^{2i}}, \\ \psi(b'_{ii}) &= \frac{b'_{ii}}{a_1^{2i}}, \\ \psi(b_{i,i+1}) &= \frac{b_{i,i+1}}{a_1^{2i+1}}, \\ \psi(b_{i+1,i}) &= \frac{b_{i+1,i}}{a_1^{2i+1}}; \end{array} \right\} \text{ and } \left\{ \begin{array}{rl} \psi(c_1) &= \frac{c_1}{a_1}, \\ \psi(c_{ii}) &= \frac{c_{ii}}{a_1^{2i}}, \\ \psi(c'_{ii}) &= \frac{c'_{ii}}{a_1^{2i}}, \\ \psi(c_{i,i+1}) &= \frac{c_{i,i+1}}{a_1^{2i+1}}, \\ \psi(c_{i+1,i}) &= \frac{c_{i+1,i}}{a_1^{2i+1}}. \end{array} \right\}$$

By dividing the equations in (4.10) by appropriate powers of a_1, it can be seen that ψ is an automorphism. Therefore, this implies $L_a \cong L_{(1,0,\dots,0)}$, completing the proof. $\square$

Proposition 4.3.5 naturally leads us to investigate the links of the vertex $(1, 0, \dots 0)$ in $D_3(q)$. The defining equations for the link of $c = (1, 0, \dots, 0) \in C$ is

$$\begin{aligned}
a_{11} + b_{11} + a_1 + a_1 b_1 + b_1 &= 0 \\
a_{12} + b_{12} + a_{11} + a_1 b_{11} &= 0 \\
a_{21} + b_{21} + a_{11} b_1 + b_{11} &= 0 \\
&\vdots \\
a_{ii} + b_{ii} + a_{i-1,i} b_1 + b_{i-1,i} &= 0 \\
a'_{ii} + b'_{ii} + a_{i,i-1} + a_1 b_{i,i-1} &= 0 \\
a_{i,i+1} + b_{i,i+1} + a_{ii} + a_1 b_{ii} &= 0 \\
a_{i+1,i} + b_{i+1,i} + a'_{ii} b_1 + b'_{ii} &= 0.
\end{aligned} \tag{4.11}$$

We can reduce this further by replacing a_1 with $a_1 + 1$ and b_1 with $b_1 + 1$. Noting that $(a_1 + 1) + (a_1 + 1)(b_1 + 1) + (b_1 + 1) = a_1 b_1 - a_1 - b_1$ in characteristic 3, we get a new set of equations, namely (4.13). We call this new series of graphs $D'(k, q)$, and take a closer look at them in the next subsection.

4.3.2 The bipartite graphs $D'(k, q)$

We now take a detour into the series of graphs $D'(k, q)$. It is worth clarifying that in this subsection, we look at $\mathbb{F}_q$ of general characteristic.

Definition 4.3.7 (The bipartite graphs $D'(q)$)**.** For a prime power q, let A and B be two copies of the countably infinite dimensional vector space V over $\mathbb{F}_q$. We use the following coordinate representations for $a \in A$, $b \in B$:

$$\begin{aligned} a &= (a_1, a_{11}, a_{12}, a_{21}, a_{22}, a'_{22}, a_{23}, \ldots, a_{ii}, a'_{ii}, a_{i,i+1}, a_{i+1,i}, \ldots), \\ b &= (b_1, b_{11}, b_{12}, b_{21}, b_{22}, b'_{22}, b_{23}, \ldots, b_{ii}, b'_{ii}, b_{i,i+1}, b_{i+1,i}, \ldots), \end{aligned} \tag{4.12}$$

Let $D'(q)$ consist of vertex set $A \sqcup B$, and let us join $a \in A$ to $b \in B$ iff the following equations hold ($i \geq 2$):

$$\begin{aligned} a_{11} - a_1 + b_{11} - b_1 + a_1 b_1 &= 0 \\ a_{12} + a_{11} + b_{12} + b_{11} + a_1 b_{11} &= 0 \\ a_{21} + a_{11} + b_{21} + b_{11} + a_{11} b_1 &= 0 \\ &\vdots \\ a_{ii} + a_{i-1,i} + b_{ii} + b_{i-1,i} + a_{i-1,i} b_1 &= 0 \\ a'_{ii} + a_{i,i-1} + b'_{ii} + b_{i,i-1} + a_1 b_{i,i-1} &= 0 \\ a_{i,i+1} + a_{ii} + b_{i,i+1} + b_{ii} + a_1 b_{ii} &= 0 \\ a_{i+1,i} + a'_{ii} + b_{i+1,i} + b'_{ii} + a'_{ii} b_1 &= 0 \end{aligned} \tag{4.13}$$

Define $D'(k, q)$ to be the graph obtained by truncation of A and B to the first k coordinates in (4.12) and, the first $k - 1$ relations in (4.13).

It is natural to inquire whether $D'(k, q)$ and $D(k, q)$ are related in any way, in particular, whether they're the same graph. The answer turns out to be yes for small values of k, but no for larger k:

Theorem 4.3.8. *(a) For* $2 \leq k \leq 6$, $D'(k, q) \cong D(k, q)$.

(b) $D'(11, 3) \ncong D(11, 3)$.

Proof. First, we prove part (a).

The main idea of the proof is as follows. Observe that it is enough to show that $D'(6, q) \cong D(6, q)$, as an isomorphism $D'(6, q) \to D(6, q)$ can be restricted to fewer coordinates to give isomorphisms $D'(k,q) \to D(k,q)$ for $k \leq 5$. To demonstrate that $D'(6,q) \cong D(6,q)$, we shall define a map $x \mapsto \bar{x}$ sending $a, b \in V(D'(6,q))$ to vectors $\bar{a}, \bar{b} \in \mathbb{F}_q^6$, such that $ab \in E(D'(6,q))$ implies $\bar{a}\bar{b} \in E(D(6,q))$. By construction, this map will be linear and invertible, which would then complete the proof.

We define the map $x \mapsto \bar{x}$ as described in Table 4.2.

$a \in V(D(6,q))$	$\bar{a} \in \mathbb{F}_q^6$	$b \in V(D(6,q))$	$\bar{b} \in \mathbb{F}_q^6$
a_1	a_1	b_1	b_1
a_{11}	$a_{11} - a_1$	b_{11}	$b_{11} - b_1$
a_{12}	$a_{12} + a_1$	b_{12}	$b_{12} + b_1$
a_{21}	$a_{21} + a_1$	b_{21}	$b_{21} + b_1$
a_{22}	$a_{22} + a_{12} + a_{11} - a_1$	b_{22}	$b_{22} + b_{12} + b_{11} - b_1$
a'_{22}	$a'_{22} + a_{21} + a_{11} - a_1$	b'_{22}	$b'_{22} + b_{21} + b_{11} - b_1$

Table 4.2: The isomorphism $D'(6,q) \to D(6,q)$

Suppose $a, b \in V(D'(6,q))$ with $ab \in E(D'(6,q))$. This implies:

$$\begin{aligned}
a_{11} - a_1 + b_{11} - b_1 + a_1 b_1 &= 0 \\
a_{12} + a_{11} + b_{12} + b_{11} + a_1 b_{11} &= 0 \\
a_{21} + a_{11} + b_{21} + b_{11} + a_{11} b_1 &= 0 \\
a_{22} + a_{12} + b_{22} + b_{12} + a_{12} b_1 &= 0 \\
a'_{22} + a_{21} + b'_{22} + b_{21} + a_1 b_{21} &= 0.
\end{aligned}$$

Now observe that, $\bar{a}_1 = a_1$ and $\bar{b}_1 = b_1$. Further,

- $$\begin{cases} \bar{a}_{11} + \bar{b}_{11} + a_1 b_1 = a_{11} - a_1 + b_{11} - b_1 + a_1 b_1 \\ \qquad\qquad\qquad\quad = 0, \end{cases}$$

- $$\begin{cases} \bar{a}_{12} + \bar{b}_{12} + a_1 \bar{b}_{11} = a_{12} + a_1 + b_{12} + b_1 + a_1(b_{11} - b_1) \\ \qquad = a_{12} + a_1 + b_{12} + b_1 + a_1 b_{11} + (a_{11} - a_1 + b_{11} - b_1) \\ \qquad = a_{12} + a_{11} + b_{12} + b_{11} + a_1 b_{11} \\ \qquad = 0, \end{cases}$$

- $$\begin{cases} \bar{a}_{21} + \bar{b}_{21} + \bar{a}_{11}b_1 = a_{21} + a_1 + b_{21} + b_1 + (a_{11} - a_1)b_1 \\ \qquad = a_{21} + a_1 + b_{21} + b_1 + a_{11}b_1 + (a_{11} - a_1 + b_{11} - b_1) \\ \qquad = a_{21} + a_{11} + b_{21} + b_{11} + a_{11}b_1 \\ \qquad = 0, \end{cases}$$
- $$\begin{cases} \bar{a}_{22} + \bar{b}_{22} + \bar{a}_{12}b_1 = a_{22} + a_{12} + a_{11} - a_1 + b_{22} + b_{12} + b_{11} - b_1 + (a_{12} + a_1)b_1 \\ \qquad = a_{22} + a_{12} + b_{22} + b_{12} + a_{12}b_1 \\ \qquad = 0, \end{cases}$$
- $$\begin{cases} \bar{a}'_{22} + \bar{b}'_{22} + \bar{a}_1\bar{b}_{21} = a'_{22} + a_{21} + a_{11} - a_1 + b'_{22} + b_{21} + b_{11} - b_1 + a_1(a_{21} + b_1) \\ \qquad = a'_{22} + a_{21} + b'_{22} + b_{21} + a_1b_{21} \\ \qquad = 0. \end{cases}$$

Therefore the map $x \mapsto \bar{x}$ is an isomorphism from $D'(6,q)$ to $D(6,q)$, as desired. ■

Our proof of part (b) is purely computational. In summary, it has been computed that the diameter of the component of $D(11,3)$ containing $\vec{0}$ is 22 whereas the same number for $D'(11,3)$ is 20, implying they're not isomorphic (as it is known that $D(11,3)$ is edge-transitive). Further, $D(11,3)$ has 112 cycles through the edge $\{\vec{0},\vec{0}\}$ whereas $D'(11,3)$ has only 4. This also implies $D(11,3) \ncong D'(11,3)$.

The github repository in the following link contains further details on how to reproduce these results: `https://github.com/Potla1995/hypergraphSuspension/` □

Remark. Computer calculations for small values of q suggest that $D'(k,q)$ and $D(k,q)$ are isomorphic for $7 \leq k \leq 10$. However, the proof method used for $k \leq 6$ does not extend to this range.

Note that proving that $D'(k,q)$ has high girth is synonymous to proving lower bounds on $\mathrm{ex}(n, \widetilde{C}_{2k})$ by the machinery we've built so far in this section, and we believe there is enough evidence, computational, and otherwise, to make the following conjecture, analogous to $D(k,q)$.

Conjecture 4.3.9. $D'(k,q)$ has girth at least $k+4$ if k is even, and $k+5$ if k is odd.

4.3.3 Proof of Theorem 4.2.2

We have now built all the machinery required to complete our proof of Theorem 4.2.2, and will delve into the proof.

Proof of Theorem 4.2.2. Recall that we have to check three properties of $D_3(k,q)$, and that q is a power of 3.

1. First, we check that $D_3(k,q)$ has $3q^k$ vertices and q^{2k+1} edges. It is clear that every part of $D_3(k,q)$ has q^k vertices. Since there is exactly one free variable when we fix a and b for a hyperedge $\{a,b,c\}$, this gives us a total of $q^k \cdot q^k \cdot q = q^{2k+1}$ edges.

2. Next, we shall prove that the link graphs of every vertex of $D_3(k,q)$ is isomorphic, in fact, to $D(k,q)$ for $k \leq 6$. By Proposition 4.3.5, the link of every vertex of $D_3(k,q)$ is isomorphic to $D(k,q)$ or $D'(k,q)$ as $3 \mid q$. However, $D(k,q) \cong D'(k,q)$ for $k \leq 6$, implying the required assertion.

3. Finally, it remains to show that $D_3(3,q)$ is $\widetilde{C}_6$-free and $D_3(5,q)$ is $\widetilde{C}_8$-free. From the previous point, and since $D(3,q)$ and $D(5,q)$ are known to have girths 8 and 10 respectively (Proposition 4.3.2 pt. 2), this completes the proof.

□

4.4 The arc construction and Wenger's construction

In this section, we relate the arc construction and Wenger's construction via Proposition 4.2.4, and provide a new set of C_6-free graphs with n vertices and $\Theta(n^{4/3})$ edges via proving Theorem 4.2.5.

4.4.1 Proof of Proposition 4.2.4

Our main goal is to algebraically parametrize the constructions $G_{\mathtt{arc}}(k, q, \alpha_0)$ for $k \geq 2$, prime powers q and the normal rational curve α_0, which would lead us to Wenger's construction $H(k, q)$. To this end, we would require the use of the Plücker embedding [37], an algebraic geometric tool that allows us to parametrize the set L.

Lemma 4.4.1 (Plücker Embedding)**.** *Every line ℓ passing through points $[a_1 : \cdots : a_{t+1}]$ and $[b_1 : \cdots : b_{t+1}]$ in $PG(t, q)$ can be parametrized using $\binom{t+1}{2}$ coordinates $\{w_{ij} : 1 \leq i < j \leq t+1\}$, where w_{ij} is given by the i, j'th minor of the matrix*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{t+1} \\ b_1 & b_2 & \cdots & b_{t+1} \end{bmatrix}.$$

For further details on the Plücker embedding, the reader is referred to [37], p.211.

We are now well-equipped to delve into the proof of Proposition 4.2.4, which asserts that $G_{\mathtt{arc}}(k, q, \alpha_0^-) \cong H(k, q)$.

Proof of Proposition 4.2.4. Recall that in the $G_{\mathtt{arc}}(k, q, \alpha_0^-)$ construction, $P = \Sigma \setminus \Sigma_0$ and

$$L = \{\text{projective lines } \ell : \ell \cap \Sigma_0 \in \alpha_0^-\}.$$

Therefore, $|P| = q^k$ and $|Q| = q^{k-1}|\alpha_0^-| = q^k$.

Observe that the lines in L pass through the points $[1 : a_1 : \cdots : a_k] \in P$ and $[0 : 1 : x : \cdots : x^{k-1}] \in \alpha_0$. Let $\{w_{ij} : 1 \leq i < j \leq k+1\}$ parametrize lines in L. Then, for $2 \leq j \leq k+1$,

$$w_{1j} = \det \begin{bmatrix} 1 & a_{j-1} \\ 0 & x^{j-2} \end{bmatrix} = x^{j-2}, \tag{4.14}$$

and for $2 \leq i < j$,

$$w_{ij} = \det \begin{bmatrix} a_{i-1} & a_{j-1} \\ x^{i-2} & x^{j-2} \end{bmatrix} = x^{i-2}(a_{i-1}x^{j-i} - a_{j-1}). \tag{4.15}$$

This set of relations imply

$$x = w_{13}; w_{1j} = w_{13}^{j-2}, 2 \leq j; \text{ and } w_{ij} = w_{13}^{i-2}(w_{2j} - w_{2,j-1}w_{13}), 2 \leq i < j \leq k+1. \tag{4.16}$$

As w_{1j} are all dependent on w_{13} and $\{w_{ij} : i \geq 3\}$ are all dependent on $\{w_{2j} : j \geq 3\}$ by (4.16), we may reduce our variables to only the set $\{w_{13}\} \cup \{w_{2j} : 3 \leq j \leq k+1\}$. Let $b_1 := x = w_{13}$ and $b_{j-1} = w_{2j}, 3 \leq j \leq k+1$. Then, the equation (4.15) for $i = 2$ reduces to

$$b_{j-1} = a_1 b_1^{j-2} - a_{j-1}, \ 3 \leq j \leq k+1,$$

Which is exactly the defining set of equations for the graph $H(k,q)$. As P consists of q^k points parametrized by $\{w_{13}\} \cup \{w_{2j} : 3 \leq j \leq k+1\}$, this implies $G_{\texttt{arc}}(k, q, \alpha_0^-) \cong H(k,q)$. $\square$

4.4.2 Proof of Theorem 4.2.5

We remark that Theorem 4.2.5 can be proved completely analogously to the proof of Proposition 4.2.4 via using the arc α of $PG(2, 2^r)$ given by $\alpha = \{[1 : t : t^{2^s}] : t \in \mathbb{F}_q\}$. However, for the sake of simplicity, we provide an alternative and more direct proof following Wenger's proof in [72]. Recall that $q = 2^r$, $(s, r) = 1$, and $G(2^r, s)$ is the bipartite graph with parts $A = B = \mathbb{F}_q^3$ such that $(a_1, a_2, a_3) \in A$ and $(b_1, b_2, b_3) \in B$ are adjacent iff

$$b_2 + a_2 = a_1 b_1 \text{ and } b_3 + a_3 = a_1 b_1^{2^s}.$$

Proof of Theorem 4.2.5. Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), \ldots, f = (f_1, f_2, f_3)$ form a C_6 in $G(2^r, s)$ where $a, c, e \in A$ are distinct, and $b, d, f \in B$ are distinct.

Then, as ab and bc are edges, we have $a_2+b_2 = a_1b_1, c_2+b_2 = c_1b_1$ implying $a_2+c_2 = b_1(a_1+c_1)$ (due to characteristic 2). Similarly, $a_3+c_3 = b_1^{2^s}(a_1+c_1)$. We can write these equations as,

$$\begin{bmatrix} a_1+c_1 \\ a_2+c_2 \\ a_3+c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ b_1 \\ b_1^{2^s} \end{bmatrix} \cdot (a_1+c_1).$$

As similarly

$$\begin{bmatrix} c_1+e_1 \\ c_2+e_2 \\ c_3+e_3 \end{bmatrix} = \begin{bmatrix} 1 \\ d_1 \\ d_1^{2^s} \end{bmatrix} \cdot (c_1+e_1) \text{ and } \begin{bmatrix} e_1+a_1 \\ e_2+a_2 \\ e_3+a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ f_1 \\ f_1^{2^s} \end{bmatrix} \cdot (e_1+a_1),$$

Adding these up and using characteristic 2, we have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ b_1 \\ b_1^{2^s} \end{bmatrix} \cdot (a_1+c_1) + \begin{bmatrix} 1 \\ d_1 \\ d_1^{2^s} \end{bmatrix} \cdot (c_1+e_1) + \begin{bmatrix} 1 \\ f_1 \\ f_1^{2^s} \end{bmatrix} \cdot (e_1+a_1) \\ &= \begin{bmatrix} 1 & 1 & 1 \\ b_1 & d_1 & f_1 \\ b_1^{2^s} & d_1^{2^s} & f_1^{2^s} \end{bmatrix} \begin{bmatrix} a_1+c_1 \\ c_1+e_1 \\ e_1+a_1 \end{bmatrix}. \end{aligned} \tag{4.17}$$

Let $M(x,y,z) = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^{2^s} & y^{2^s} & z^{2^s} \end{bmatrix}$. We shall now show that if $x,y,z \in \mathbb{F}_q$ are all distinct, then $M(x,y,z)$ is invertible, i.e. $\frac{x^{2^s}+y^{2^s}}{x+y} \neq \frac{y^{2^s}+z^{2^s}}{y+z}$. To prove this, it is enough to check that for a fixed $t \in \mathbb{F}_q$,

$$\left| \left\{ \frac{(x+t)^{2^s}+t^{2^s}}{x} : x \in \mathbb{F}_q \setminus \{t\} \right\} \right| = q-1.$$

Observe that, by the binomial theorem and using the fact that $\binom{2^s}{i}$ is even for every $0 < i < 2^s$, $\frac{(x+t)^{2^s}+t^{2^s}}{x} = x^{2^s-1}$. Hence, it suffices to show that the map $x \mapsto x^{2^s-1}$ is a permutation of $\mathbb{F}_q$. However, as the multiplicative group $\mathbb{F}_q^*$ has order $q-1$, this happens only when $(2^s-1, q-1) = 1$, which is true since

$$(2^s - 1, 2^r - 1) = 2^{(s,r)} - 1 = 1$$

by assumption.

Further, note that if $b_1 = d_1$, then, as

$$b_2 + c_2 = b_1 c_1 = c_1 d_1 = c_2 + d_2$$

and

$$b_3 + c_3 = b_1^{2^s} c_1 = c_1 d_1^{2^s} = c_3 + d_3,$$

we would obtain $b = d$, a contradiction. Thus, b_1, d_1, f_1 are pairwise distinct, and therefore $M(b_1, d_1, f_1)$ is invertible. Hence, (4.17) implies

$$a_1 + c_1 = c_1 + e_1 = e_1 + a_1 = 0,$$

i.e., $a_1 = c_1 = e_1$. However, as

$$a_2 + b_2 = a_1 b_1 = c_1 b_1 = b_2 + c_2$$

and

$$a_3 + b_3 = a_1 b_1^{2^s} = c_1 b_1^{2^s} = b_3 + c_3,$$

this would imply $a = c$, a contradiction. □

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Chapter 5

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Appendix A

BINOMIAL IDENTITIES USED IN CHAPTER 2

Our goal in this section is to prove the matrix identity asserted in Proposition A.2. Recall that the binomial coefficient $\binom{-a}{s}$ is interpreted as $(-1)^s\binom{a+s-1}{s}$. Observe that with this definition, the generalized binomial coefficients also satisfy Pascal's identity $\binom{a}{s} = \binom{a-1}{s} + \binom{a-1}{s-1}$. Before seeing the proof of Proposition A.2, we establish a useful identity in Lemma A.1.

Lemma A.1. For integers $x \geq 0, y \geq z \geq 0$, we have

$$\sum_{t=0}^{z}(-1)^t\binom{x}{t}\binom{y-t}{z-t} = (-1)^z\binom{x-y+z-1}{z}. \tag{A.1}$$

Proof of Lemma A.1. One can prove this identity using induction on y. Note that when $y = z$, the identity becomes

$$\sum_{t=0}^{z}(-1)^t\binom{x}{t} = (-1)^z\binom{x-1}{z},$$

which follows from applying Pascal's identity $\binom{x}{t} = \binom{x-1}{t} + \binom{x-1}{t-1}$ to each term and telescoping.

Now suppose that (A.1) holds for some y. Then,

$$\sum_{t=0}^{z}(-1)^t\binom{x}{t}\binom{y-t+1}{z-t} = \sum_{t=0}^{z}(-1)^t\binom{x}{t}\binom{y-t}{z-t} + \sum_{t=0}^{z-1}(-1)^t\binom{x}{t}\binom{y-t}{z-t-1}.$$

By induction hypothesis, the first term is $(-1)^z\binom{x-y+z-1}{z}$ and the second term is $(-1)^{z-1}\binom{x-y+z-2}{z-1}$. Their sum is $(-1)^z\binom{x-y+z-2}{z}$, as desired. ■

We are now going to state and prove Proposition A.2. Recall the following notation: $a_{ij}^{(m)} = \binom{m-i}{j-i}$, $b_{ij}^{(m)} = (-1)^{j-i}\binom{m-i}{j-i}$, $w_{ij}^{(m)} = (-1)^{j-i}\binom{m-k+j-i-1}{j-i}$,

$$A_{k,m} = (a_{ij}^{(m)})_{1\le i,j\le k},\ B_{k,m} = (b_{ij}^{(m)})_{1\le i,j\le k}, W_{k-1,m} = (w_{ij}^{(m)})_{1\le i,j\le k-1},$$

and,

$$D_{k-1,m} = \begin{bmatrix} A_{k-1,m} & \vec{\mathbf{1}} \\ \vec{\mathbf{0}}^\intercal & 1 \end{bmatrix},\ W'_{k-1,m} = \begin{bmatrix} W_{k-1,m} & \vec{\mathbf{0}} \\ \vec{\mathbf{0}}^\intercal & 1 \end{bmatrix}.$$

Proposition A.2.

$$B_{k,k} \cdot D_{k-1,m} \cdot W'_{k-1,m} = I_k.$$

Proof. Note that $B_{k,k} = \begin{bmatrix} B_{k-1,k} & \vec{v} \\ \vec{\mathbf{0}}^\intercal & 1 \end{bmatrix}$, where $v_i = (-1)^{k-i}$, and therefore

$$B_{k,k}D_{k-1,m}W'_{k-1,m} = \begin{bmatrix} B_{k-1,k}A_{k-1,m}W_{k-1,m} & B_{k-1,k}\vec{\mathbf{1}} + v \\ \vec{\mathbf{0}}^\intercal & 1 \end{bmatrix}.$$

We verify that $B_{k-1,k}\vec{\mathbf{1}} + v = \vec{\mathbf{0}}$ and $B_{k-1,k}A_{k-1,m}W_{k-1,m} = I_{k-1}$ in Claims A.3 and A.4, respectively.

Claim A.3. $B_{k-1,k}\vec{\mathbf{1}} + v = \vec{\mathbf{0}}$.

Proof of Claim A.3. Note that the i'th row of $B_{k-1,k}\vec{\mathbf{1}}$ is

$$\sum_{j=1}^{k-1} b_{ij}^{(k)} = \sum_{j=i}^{k-1}(-1)^{j-i}\binom{k-i}{j-i} = \sum_{j=0}^{k-i-1}(-1)^j\binom{k-i}{j} = 0 - (-1)^{k-i} = -v_i,$$

as desired. ■

Claim A.4. $B_{k-1,k}A_{k-1,m}W_{k-1,m} = I_{k-1}$.

Proof of Claim A.4. Note that the (i,j)th entry of the product matrix is given by

$$\begin{aligned}
&\sum_{r=1}^{k-1}\sum_{s=1}^{k-1} b_{ir}^{(k)} a_{rs}^{(m)} w_{sj}^{(m)} \\
=&\sum_{r=1}^{k-1}\sum_{s=1}^{k-1}(-1)^{r-i+j-s}\binom{k-i}{r-i}\binom{m-r}{s-r}\binom{m-k+j-s-1}{j-s} \qquad \text{(A.2)}\\
=&\sum_{s=1}^{k-1}(-1)^{j-s}\binom{m-k+j-s-1}{j-s}\sum_{r=1}^{k-1}(-1)^{r-i}\binom{k-i}{r-i}\binom{m-r}{s-r}.
\end{aligned}$$

Observe that, using Lemma A.1 for $x = k-i, y = m-i, z = s-i$, we get

$$\begin{aligned}
\sum_{r=1}^{k-1}(-1)^{r-i}\binom{k-i}{r-i}\binom{m-r}{s-r} &= \sum_{r=i}^{s}(-1)^{r-i}\binom{k-i}{r-i}\binom{m-r}{s-r} \\
&= \sum_{r=0}^{s-i}(-1)^{r}\binom{k-i}{r}\binom{m-i-r}{s-i-r} \\
&= (-1)^{s-i}\binom{k-m+s-i-1}{s-i}.
\end{aligned}$$

Plugging this back into (A.2), we get that the (i,j)th entry of the product matrix is

$$\sum_{s=1}^{k-1}(-1)^{j-i}\binom{m-k+j-s-1}{j-s}\binom{k-m+s-i-1}{s-i} \qquad \text{(A.3)}$$

Notice that the sum in (A.3) only runs from $s = i$ to $s = j$, and therefore after the change of variable $s \mapsto s+i$, the expression reduces to

$$(-1)^{j-i}\sum_{s=0}^{j-i}\binom{m-k+j-s-i-1}{j-i-s}\binom{k-m+s-1}{s}. \qquad \text{(A.4)}$$

Note that $\binom{s-(m-k)-1}{s} = (-1)^s\binom{m-k}{s}$, so (A.4) is the sum

$$(-1)^{j-i}\sum_{s=0}^{j-i}(-1)^s\binom{m-k}{s}\binom{m-k+j-i-1-s}{j-i-s},$$

which, on invoking Lemma A.1 for $x = m-k, y = m-k+j-i-1, z = j-i$, reduces to

$$(-1)^{j-i}\cdot(-1)^{j-i}\cdot\binom{m-k-m+k-j+i+1+j-i-1}{j-i} = \binom{0}{j-i}.$$

Clearly, this is 0 when $j \neq i$ and 1 when $j = i$. ■

This completes the proof of Proposition A.2. □

Appendix B

ADDITIONAL DETAILS FOR CHAPTER 3

Our goal in this section is to complete the proof of Lemma 3.6.9. Recall that H is a subgraph of a $\{K_6^-, \widehat{P}_5\}$-free graph G such that every edge of H has codegree at least 2.

Proof of Lemma 3.6.9. We wish to show that H does not contain copies of W_5 or $K_{1,2,2}$. We do this via sequentially cleaning the following graphs from H:

- $K_6^{-2,1}$, the graph obtained from K_6 by deleting two intersecting edges,
- $K_6^{-2,2}$, the graph obtained from K_6 by deleting two non-intersecting edges,
- $K_6^{-3,1}$, the graph obtained from K_6 by deleting a P_3,
- $K_6^{-3,2}$, the graph obtained from K_6 by deleting a $P_2 \sqcup K_2$,
- K_5,
- K_5^-, the graph obtained from K_5 by deleting one edge,
- W_5^+, the graph obtained from the 5-wheel graph $W_5 = \widehat{C}_5$ by adding an edge,

- W_5, and
- $K_{1,2,2}$, the 4-wheel graph.

More specifically, whenever H contains a copy of one of these graphs, we would be able to use the induction hypothesis of (3.17) on some subgraph $H' \subsetneq H$ and complete the induction step for H.

Before proceeding onto the cleaning steps, we make an important observation:

$$\text{If } abcde \text{ is a } P_4 \text{ in } N_H(x), \text{ then } N_H(\{a,x\}) \subseteq \{b,c,d,e\}. \tag{B.1}$$

This is because since xa has codegree at least 2, we must have a vertex $y \in V(H)$ with xay being a triangle, $y \neq b$. If $y \notin \{c,d,e\}$, then we get a $\widehat{P}_5$ centered around x. Thus $y = c$ or $y = d$ or $y = e$, implying (B.1).

1. Cleaning $K_6^{-2,1}$: Suppose H has a copy of $K_6^{-2,1}$ with vertices $\{a,b,c,d,e,f\}$ such that the edges ab and bc are missing. This is an induced subgraph as H has no K_6^-. Further, all edges of this subgraph other than ac cannot belong to external triangles, as verified in Figure B-1.

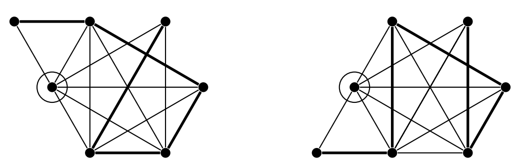

Figure B-1: All edges but ac cannot lie in external triangles.

Now suppose ac lies on an external triangle, acx. By (B.1) on the 4-path $fedcx$ in $N_H(a)$, $N_H(\{a,x\}) \subseteq \{c,d,e\}$. Moreover, ax has codegree at least 2. Thus, xd, xe, or xf is an edge of H. In either case we obtain $\widehat{P}_5$'s in H, as shown in Figure B-2.

Hence $K_6^{-2,1}$ is a block by itself. Let H' be the subgraph of H obtained by deleting all edges from this copy of $K_6^{-2,1}$. Then note that $t(H') = t(H) - 13$ and $e(H') = e(H) - 13$, completing the induction step for H.

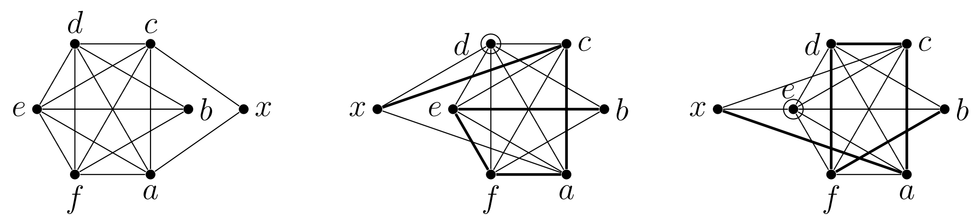

Figure B-2: acx is a triangle.

2. Cleaning $K_6^{-2,2}$: Let H have a copy of $K_6^{-2,2}$ with vertices $\{a, b, c, d, e, f\}$ such that the edges ab and cd are missing. Clearly this is an induced subgraph of H. It can be checked that the edges ea, ec, eb, ed cannot lie on external triangles as otherwise we would get $\widehat{P}_5$'s centered at e. Similarly, the edges fc, fa, fd, fb cannot lie on external triangles.

Now, suppose the edge ad lies in an external triangle adx. Refer to Figure B-3. As $cfedx$ is a P_4 in the neighborhood of a, (B.1) implies that $N_H(\{a, x\}) \subseteq \{d, e, f, c\}$ and $N_H(\{a, c\}) \subseteq \{f, e, d, x\}$. If $xe \in E(H)$, this leads to a $\widehat{P}_5$ centered at e, given by the 5-path $caxdbf$. If $xf \in E(H)$, we get a $\widehat{P}_5$ centered at f, given by the 5-path $bdxac$. Thus, $xc \in E(H)$, and the edge xa is light. Repeating the same argument for the 4-path $bfeax$ around d, we get $xb \in E(H)$, xd is light and ac has codegree 3. Using (B.1) on the 4-path $xaefb$ in $N_H(c)$ and $N_H(d)$ respectively, we get $N_H(\{c, x\}) \subseteq \{a, e, f, b\}$ and $N_H(\{b, d\}) \subseteq \{f, e, a, x\}$. Since we already know that $xe, xf, ab \notin E(H)$, this means that the edge xc is light and bd has codegree 3. Similarly, bx is light. Now, let

$$H' = H - \{ea, ec, eb, ed, fc, fa, fd, fb, xa, xc, xb, xd, ac, bd\}.$$

Clearly $e(H') = e(H) - 14$ and $t(H') = t(H) - 14$, and we are done by induction.

Therefore, ad cannot lie in any external triangle adx, and is light. Similarly, bc is light. Let $H' = H - \{ea, ec, eb, ed, fc, fa, fd, fb, ad, bc\}$. Then $t(H') = t(H) - 10$ and $e(H') = e(H) - 10$, finishing the induction step for H.

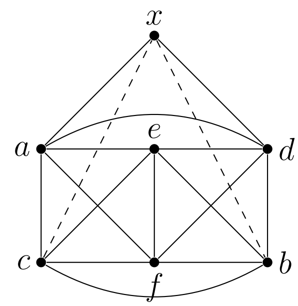

Figure B-3: Edge ad lies in triangle adx, $x \notin \{a, b, c, d, e, f\}$

3. Cleaning $K_6^{-3,1}$: Since G has no $K_6^{-2,1}$ or $K_6^{-2,2}$ which are the only two ways one can delete two edges from K_6, any copy of $K_6^{-3,1}$ is induced. Suppose such a copy of $K_6^{-3,1}$ exists in G, and is given by the complete graph on $\{a, b, c, d, e, f\}$ minus the edges $\{ab, bc, cd\}$. By an argument exactly the same as before, $ea, ec, eb, ed, fc, fa, fd$, fb are light. Further, if ad lies in an external triangle adx (as in Figure B-4), then by repeating the argument for cleaning $K_6^{-2,2}$, we note that $xc, xb \in E(H)$, xa, xd are light, and ac, bd have codegree exactly three. Also, by using (B.1) on the 4-path $dxcfe$, we have $N_H(\{a, d\}) \subseteq \{x, c, f, e\}$. As $cd \notin E(H)$, this means $N_H(\{a, d\}) = \{x, f, e\}$, and therefore ad has codegree three. Finally, (B.1) on the path $cfedx$ in $N_H(a)$ gives us $N_H(\{a, c\}) \subseteq \{f, e, d, x\}$, whereas $cd \notin E(H)$, implying that ac has codegree three as well. Similarly, bd has codegree three.

Let $H' = H - \{ea, ec, eb, ed, fc, fa, fd, fb, ac, bd, ad, xa, xd\}$. Then, $t(H') = t(H) - 13$ and $e(H') = e(H) - 13$, and we can proceed by the induction hypothesis on H'.

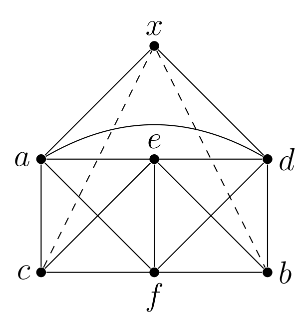

Figure B-4: adx is an external triangle, ab, bc, ca are non-edges in H.

On the other hand, if the edge ad is light, then we can simply let $H' = H -$

$\{ea, ec, eb, ed, fc, fa, fd, fb, ad\}$, whence $t(H') = t(H) - 9$ and $e(H') = e(H) - 9$, and the induction step would be complete. Hence we can assume that H is $K_6^{-3,1}$-free.

4. Cleaning $K_6^{-3,2}$: Suppose H contains a $K_6^{-3,2}$ on vertices $\{a, b, c, d, e, f\}$ such that edges ab, cd, de are missing. Since H is $K_6^{-2,1}$ and $K_6^{-2,2}$-free, this subgraph is induced. As the edges bd and ad must have codegree at least two, there exist $x, y \in V(H) \setminus \{a, b, c, d, e, f\}$ such that bdx and ady are triangles in H. We consider two different cases.

- **Case 1.** $x = y$ (Figure B-5 (left)): Since $N_H(b)$ contains the 4-path $cefdx$, (B.1) gives $N_H(\{b, x\}) \subseteq \{d, f, e, c\}$. If $xf \in E(H)$, then $N_H(f)$ contains the 5-path $xadbce$. If both xc and xe were edges in H, then $H[\{a, b, c, e, f, x\}]$ would be a $K_6^{-2,2}$ with edges xf, ab missing. Therefore, only one of xc and xe can be an edge. By symmetry, assume $xc \in E(H)$ and $xe \notin E(H)$.

  As this fixes edges and non-edges between any pair of vertices from the 7-set $\{a, b, c, d, e, f, x\}$, $H[\{a, b, c, d, e, f, x\}]$ is induced. Consider the 5-wheel $(f, adbcea)$, where the first tuple denotes the central vertex and the second tuple is the outer C_5. Since none of the edges fa, fb, fc, fd, fe can lie in triangles with a vertex $y \notin \{a, b, c, d, e, f, x\}$ (it would give a $\widehat{P}_5$ around f), they all have exhausted their codegrees. Similarly, $(c, befaxb)$, $(b, cxdfec)$, and $(a, dxcefd)$ are W_5's in H. Let

  $$H' = H - \{cb, cx, ca, cf, ce, fe, fb, fd, fa, be, bx, bd, ad, ax, ae\}.$$

  Then, $e(H') = e(H) - 15$ and $t(H') = t(H) - 13$ (4 triangles through x, 7 through f but not x, and 2 not through x or f). We can then proceed with the induction hypothesis on H'.

- **Case 2.** $x \neq y$ (Figure B-5 (right)): Without loss of generality assume $by, ax \notin E(H)$, as these would lead us to Case 1. As $N_H(b)$ contains the 4-path $xdfec$, by

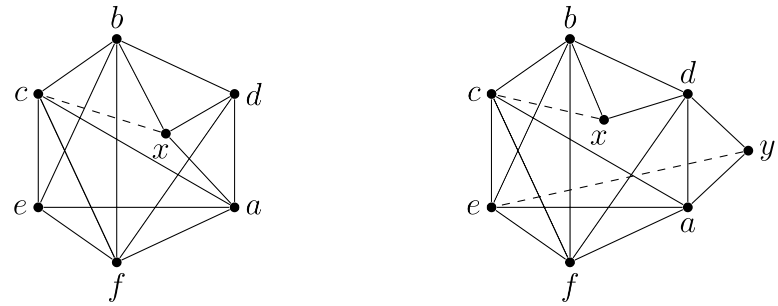

Figure B-5: bdx and ady are triangles, (left: $x = y$, right: $x \neq y$)

(B.1), $N_H(\{b, x\}) \subseteq \{d, f, e, c\}$. Note that if $xf \in E(H)$, then $N_H(f)$ contains the path $ecbxda$ of length 5. Hence, $N_H(\{b, x\}) \subseteq \{d, e, c\}$. Further, both xc and xe cannot be edges in H, as then $H[\{a, b, c, e, f, x\}] \supseteq K_6^{-3,1}$ with the edges ba, ax, xf missing. As codegree of bx is at least 2, exactly one of xc and xe is an edge in H. By symmetry, assume $xc \in E(H)$ and $xe \notin E(H)$.

Now by (B.1) on the 4-path $ydfec$ in $N_H(a)$, we get $N_H(\{a, y\}) \subseteq \{d, f, e, c\}$. If $yf \in E(H)$, then $N_H(f)$ contains the path $aydbce$ of length 5, and if $yc \in E(H)$, then $N_H(c)$ contains the path $xbefay$ of length 5. Therefore, $N_H(\{a, y\}) = \{d, e\}$, and $ye \in E(H)$.

Finally, let us consider the 4-path $yafbc$ in $N_H(e)$. Using (B.1),

$$N_H(\{y, e\}) \subseteq \{a, f, b, c\}.$$

However, $yf, yc \notin E(H)$ from our argument in the last paragraph, and $by \notin E(H)$ as we are in Case 2. This is a contradiction, as the edge ye must have codegree at least 2.

5. Cleaning K_5: If H contains a copy of K_5 on vertex set $\{a, b, c, d, e\}$, then we claim that it is a block by itself. Suppose $x \in V(H) \setminus \{a, b, c, d, e\}$ is such that abx is a triangle in H. Since $xbcde$ is a P_4 in $N_H(a)$, (B.1) implies that $N_H(\{a, x\}) \subseteq \{b, c, d, e\}$. Further, ax must have codegree at least 2. Thus, xc, xd, or xe is an edge. In either case, $H[\{a, b, c, d, e, x\}] \supseteq K_6^{-2,1}$, a contradiction.

6. Cleaning K_5^-: Let H have a copy of K_5^- on vertices a, b, c, d, e such that $ab \notin E(H)$. If the edge bc lies in an external triangle bcx as shown in Figure B-6, then note that xb has codegree at least two, and (B.1) on the 4-path $xbdea$ in $N_H(c)$ tells us that $N_H(\{c, x\}) \subseteq \{b, d, e, a\}$. If $xe \in E(H)$ then $G[a, b, c, d, e, x]$ contains the graph $K_6^{-3,1}$ with edges dx, xa, ab missing. If $xd \in E(H)$, then we have the $K_6^{-3,1}$ with edges ex, xa, ab missing. Finally, if $xa \in E(H)$, then $G[a, b, c, d, e, x]$ contains $K_6^{-3,2}$ with edges ex, xd, ab missing. Thus, the edge bc cannot lie on an external triangle.

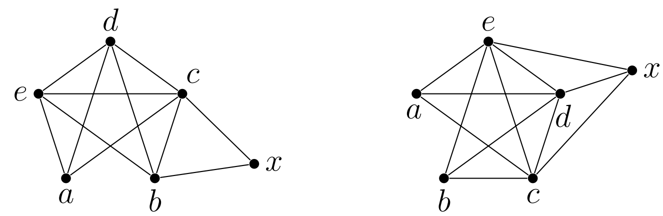

Figure B-6: K_5^- is a block by itself.

Thus by symmetry, ae, ad, ac, be, bd, bc cannot lie on external triangles. Now suppose that the edge cd lies on an external triangle cdx. By (B.1) on any $\widehat{P}_4$ centered at c, $N_H(\{c, x\}) \subseteq \{d, e, a, b\}$. If either xa or xb is an edge, we obtain a $K_6^{-3,1}$ with missing edges ab, bx, xe or ba, ax, xe, respectively. So assume $xa, xb \notin E(H)$. Thus $ex \in E(H)$, and cx has codegree 2. Similarly, dx has codegree 2. Now using (B.1) on the 4-path $xdbca$ in $N_H(e)$, we have $N_H(\{e, x\}) \subseteq \{a, b, c, d\}$. Since $xa, xb \notin E(H)$, ex must have codegree 2. Thus, the edges xc, xd, xe all have codegree 2. Let $H' = H - \{xc, xd, xe\}$, then $t(H') = t(H) - 3$ and $e(H') = e(H) - 3$, and we can proceed by induction.

Hence, we may assume that cd also does not lie on external triangles. Let $H' = H - \{ac, ad, ae, bc, bd, be, cd\}$. Then, $t(H') = t(H) - 7$ and $e(H') = e(H) - 7$, completing the induction hypothesis for H again. So, without loss of generality we can assume that G is $\{K_5^-, \widehat{P}_5\}$-free.

7. Cleaning W_5^+: Let H contain a W_5^+, given by central vertex x, outer cycle $abcde$ with an edge $ac \in E(H)$. If $ad \in E(H)$, then a, b, c, d, x form a K_5^- in H. Therefore by symmetry, all copies of W_5^+ in H are induced.

Now let us fix such a W_5^+ in H with vertices labeled as above. As cd and ae must have codegree at least 2, there exist $y, z \in V(H) \setminus \{a, b, c, d, e, x\}$ such that aey and cdz are triangles in H. Then, we have two possible cases:

- **Case 1:** $y = z$. Refer to Figure B-7. Note that if $yx \in E(H)$, then $N_H(x)$ would contain the P_5 given by $yabcde$. Hence $yx \notin E(H)$. Using (B.1) on the 4-path $yexcb$, $N_H(\{a, y\}) \subseteq \{e, x, c, b\}$. Suppose $yb \in E(H)$. Note that $(y, abcdea)$, $(c, aydxba)$, $(a, cybxec)$ form W_5^+'s in H. As (B.1) together with the fact that every W_5^+ of H is induced imply that each central edge of any copy of W_5^+ in H does not lie on external triangles, the edges xa, xb, xc, xd, xe; ya, yb, yc, yd, ye; cb, ca, cd; ab, ae cannot lie in external triangles. Thus, we can delete these 15 edges from H, and only lose 13 triangles (6 through x, 6 through y, and the triangle abc). Our proof would then be complete by induction.

  Hence, assume $yb \notin E(H)$. From $N_H(\{a, y\}) \subseteq \{e, x, c, b\}$ this implies that codegree of ay is exactly 2. Similarly, cy has codegree 2. Further, (B.1) on the path of length four $bxcye$ in $N_H(a)$ implies that $N_H(\{a, b\}) \subseteq \{x, c, y, e\}$. Since $by, be \notin E(H)$, this implies that $N_H(\{a, b\}) = \{x, c\}$, and ab has codegree 2. Similarly, bc has codegree 2. Let $H' = H - \{xa, xb, xc, xd, xe, ab, bc\}$. Then, $t(H') = t(H) - 7$ and $e(H') = e(H) - 7$, again concluding the induction step.

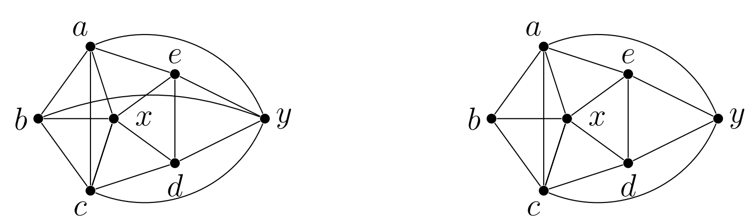

Figure B-7: Case 1: $y = z$, aey and cdy are triangles in H.

- **Case 2:** $y \neq z$. Since $N_H(a)$ contains the path $yexcb$ of length 4, we must have $N_H(\{a, y\}) \subseteq \{e, x, b, c\}$. If $yx \in E(H)$, we obtain the 5-path $yabcde$ in $N_H(x)$, and if $yc \in E(H)$ we obtain the 5-path $yabxdz$ in $N_H(c)$. On the other hand,

ay must have codegree at least 2. Thus $N_H(\{a,y\}) = \{b,e\}$ and ay is light. By a symmetric argument, $zb \in E(H)$ and cz is also light. Refer to Figure B-8. Since $N_H(b)$ contains the P_4 given by $zcxay$, we have $N_H(\{b,y\}) \subseteq \{a,x,c,z\}$. However, we have already observed that $yx, yc \notin E(H)$. Hence $zy \in E(H)$, and by is light. Similarly, bz is light.

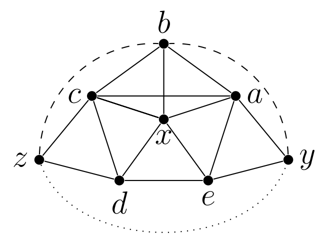

Figure B-8: Case 2: $y \neq z$, aey and cdz are triangles in H.

Observe that we have produced two W_5^+'s given by $(c, zdxabz)$ and $(a, yexcby)$, both with the extra edge ac. By (B.1) in $N_H(c)$ and $N_H(b)$, all of the central edges cannot lie in external triangles. Let

$$H' = H - \{cz, cd, cx, cb, ay, ae, ax, ab, ac, by, bz\}.$$

It is clear that $e(H') = e(H) - 11$, and that deleting these edges, we delete 6 triangles through c and 4 triangles through a that do not contain c, and the triangle byz through b which does not contain a or c. Hence, $t(H') = t(H) - 11$. This completes our induction step.

We may therefore assume that H is W_5^+-free.

8. Cleaning W_5**:** Suppose H has a copy of W_5 given by $(x, abcdea)$. As H is W_5^+-free, $H[\{a,b,c,d,e,x\}] \cong W_5$. By (B.1) applied to $N_H(x)$, every central edge is light. Thus, we may let $H' = H - \{xa, xb, xc, xd, xe\}$, whence $t(H') = t(H) - 5$ and $e(H') = e(H) - 5$, allowing us to complete the induction step for H.

9. Cleaning $K_{1,2,2}$: Finally, let H contain a $K_{1,2,2}$ with central vertex x and outer cycle $abcd$. Since H is K_5^--free, $H[\{a,b,c,d,x\}] \cong K_{1,2,2}$. We claim that none of the edges xa, xb, xc, xd lie on an external triangle.

For the sake of contradiction, assume $y \in V(H) \setminus \{a,b,c,d,x\}$ is such that xay is a triangle in H (Figure B-9). By (B.1) in $N_H(a)$, we have $N_H(\{a,y\}) \subseteq \{x,d,c,b\}$. If $yd \in E(H)$, we obtain the 5-wheel $(x, ydcbay)$, and if $yb \in E(H)$, we obtain the 5-wheel $(x, yadcby)$. Since $|N_H(\{a,y\})| \geq 2$, we must have $N_H(\{a,y\}) = \{x,c\}$. But then, $H[\{c,x,b,y,a\}] \cong K_5^-$, a contradiction.

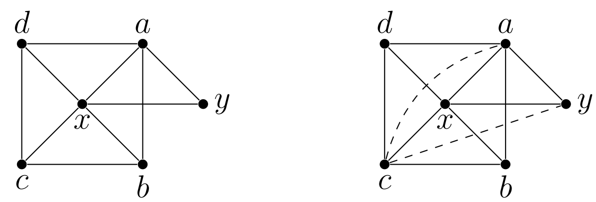

Figure B-9: xa cannot lie in an external triangle xay.

Hence, the edges xa, xb, xc, xd all have codegree 2. Let $H' = H - \{xa, xb, xc, xd\}$, then $t(H') = t(H) - 4$ and $e(H') = e(H) - 4$, finishing the induction step in this case as well.

Hence, after these cleaning procedures, we may assume that H is a $\{W_5, K_{1,2,2}\}$-free subgraph of G such that every edge of H has codegree at least 2. This concludes the proof of Lemma 3.6.9. $\square$

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