The bipartite graphs $D'(k, q)$ have low girth

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Abstract. In [3], we introduced a family of 3-uniform hypergraphs $D_3(k, q)$ extending the Lazebnik–Ustimenko–Woldar graphs $D(k, q)$. For $3 \mid q$, the link graphs of $D_3(k, q)$ include a new family of bipartite graphs $D'(k, q)$. Due to computational evidence that $D'(k, q)$ and $D(k, q)$ have the same girth for $k \leq 18$ and $q = 3$, it was hoped that the high-girth property persists for every $q = 3^n$. We show that this hope fails for every nontrivial extension of $\mathbb{F}_3$: for $q = 3^n$ with $n \geq 2$ and $k \geq 2$, the graph $D'(k, q)$ contains a cycle of length 10. In particular, the girth of $D'(k, q)$ for $k \geq 5$ is at most 10. The proof constructs two length-five walks from the zero vertex that share a common midpoint, using an involution of $\mathbb{F}_q$ and a symbolic block induction over $\mathbb{F}_3[X]$. The accompanying repository `https://github.com/Potla1995/dgraphs` contains the computations that motivated the construction and verify representative finite cases; the proof itself is algebraic.

Keywords: algebraic graphs, girth, extremal combinatorics, finite fields, hypergraph Turán problem.

1 Introduction

Constructing bipartite graphs with many edges and high girth has been a central problem in extremal combinatorics. For a graph G and integer $n \geq 1$, let $\mathrm{ex}(n, C_{2k})$ denote the maximum number of edges in an n-vertex graph containing no cycle of length $2k$. The Bondy–Simonovits theorem [1] implies $\mathrm{ex}(n, C_{2k}) = O(n^{1+1/k})$, while the best-known lower bounds are achieved by the connected components of the bipartite graphs $D(k, q)$ of Lazebnik, Ustimenko, and Woldar [2]. The graphs $D(k, q)$ of Lazebnik and Ustimenko have girth at least $k + 4$, and the connected components $CD(k, q)$ of $D(k, q)$ satisfy $\mathrm{girth}(CD(k, q)) \geq k + 5$ for odd k over $\mathbb{F}_q$ with $\gcd(q, k) = 1$.

Analogous constructions can be sought for hypergraph Turán problems. For a graph G, the *hypergraph suspension* $\widetilde{C}_{2k}$ is the 3-uniform hypergraph obtained by adding a new vertex adjacent to every edge of C_{2k}. In [3], the author constructed a family of 3-uniform hypergraphs $D_3(k, q)$ over $\mathbb{F}_q$ with $3 \mid q$, extending $D(k, q)$. That paper showed that $D_3(3, q)$ is $\widetilde{C}_6$-free and $D_3(4, q)$ is $\widetilde{C}_8$-free, yielding $\mathrm{ex}_3(n, \widetilde{C}_6) = \Theta(n^{7/3})$ and $\mathrm{ex}_3(n, \widetilde{C}_8) = \Omega(n^{11/5})$.

The key structural feature of $D_3(k, q)$ is that its vertex links are bipartite graphs of two types: $D(k, q)$ and a new family $D'(k, q)$. The two families are isomorphic for $k \leq 6$ and $3 \mid q$, but the girth analysis of $D'(k, q)$ for large k was left open. Computational data for $q = 3$ showed that $D'(k, 3)$ and $D(k, 3)$ have the same girth for all computed values of k up to $k = 18$, suggesting that $D'(k, q)$ might also have high girth for every $q = 3^n$. Further, it was noted in [3] that $D'(11, 3) \not\cong D(11, 3)$, implying that $D'(k, q)$ is indeed a new family of graphs.

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A lower bound of $k+5$ for odd k on the girth of $D'(k,q)$ for $q=3^r$ would imply $\mathrm{ex}_3(n,\widetilde{C}_{2k}) \geq \Omega(n^{2+1/(2k-3)})$, providing new lower bounds for all $k \geq 4$.

The main result of this note shows that this idea fails at the first nontrivial step:

Theorem 1.1 (Main theorem)**.** *Let $q=3^n$ with $n \geq 2$, and let $k \geq 2$. Then $D'(k,q)$ contains a cycle of length* 10. *In particular, the girth of $D'(k,q)$ is at most* 10 *for every $k \geq 5$.*

The case $q=3$ appears exceptional: computational evidence shows that $D'(k,3)$ has the same (much larger) girth as $D(k,3)$ for all $k \leq 18$. Even the smallest nontrivial extension of $\mathbb{F}_3$ already produces 10-cycles in $D'(k,q)$.

The proof idea is to explicitly construct such a 10-cycle. Fix $\theta \in \mathbb{F}_q \setminus \mathbb{F}_3$ and let $\bar{\theta} = -\theta - 1$. The involution $\lambda \mapsto -\lambda - 1$ on $\mathbb{F}_q$ sends θ to $\bar{\theta}$. We construct two length-five walks from the zero vertex of $D'(k,q)$ with first-coordinate words $(\theta, 2, 2, \bar{\theta}, 0)$ and $(\bar{\theta}, 2, 2, \theta, 0)$, and show that they share a common midpoint line. The midpoint equality follows from polynomial identities in the coordinate ring $\mathbb{F}_3[X,Y]/(X+Y+1)$: the midpoint $m(X)$, computed coordinatewise as a polynomial in X, is invariant under the involution $X \mapsto -X-1$. This invariance is proved by a block induction on the coordinates of the penultimate vertex u_4, using four auxiliary identities.

Several sanity checks for the proof, and a toolkit for analyzing algebraically defined graphs are developed in the `dgraphs` repository [4]. The file `src/check9.cpp` directly verifies the 10-cycle witness for specific (k,q) without building the full graph, the file `src/check_midpoint.cpp` verifies that the midpoint equality $m(X) = m(-X-1)$ holds for all $X \in \mathbb{F}_q$ numerically, and the Jupyter notebook at `notebooks/characteristic3_symbolic_checks.ipynb` combines all symbolic checks into a single document.

As a consequence of Theorem 1.1, the family $D_3(k,q)$ is ruled out as a uniform source of lower bounds for $\mathrm{ex}_3(n,\widetilde{C}_{2k})$ via link girth, beyond the already established cases. In particular, whether $\mathrm{ex}_3(n,\widetilde{C}_8) = \Theta(n^{11/5})$ or $\Theta(n^{9/4})$ remains unknown, and improving the lower bounds for $k \geq 5$ requires designing new graph families.

2 Preliminaries

We recall the definition of $D'(k,q)$ in the grouped coordinate form used throughout the proof.

Fix a prime power q and a positive integer k. Let P and L be two copies of $\mathbb{F}_q^k$, called *points* and *lines*. We write points as $p = (p_1, \ldots, p_k)$ and lines as $[l] = (l_1, \ldots, l_k)$.

The *grouped coordinate order* arranges the k coordinates as

$$\big(x_1, x_{11}, x_{12}, x_{21}; x_{22},\ x'_{22},\ x_{23},\ x_{32}; x_{33},\ x'_{33},\ x_{34},\ x_{43}; \ldots\big),$$

where for every $i \geq 2$ the four coordinates $(x_{ii},\ x'_{ii},\ x_{i,i+1},\ x_{i+1,i})$ form the *block of level* i, and the initial coordinates $(x_1, x_{11}, x_{12}, x_{21})$ form levels 0 and 1. [2]

[2] In flat-index notation (used in the code), the block-i coordinates occupy positions $4(i-1)$ through $4(i-1)+3$.

Definition 2.1. The bipartite graph $D'(k,q)$ has vertex set $V = P \sqcup L$. A point p and a line $[l]$ are *adjacent* if and only if they satisfy the equations

$$\begin{aligned} l_{11} + p_{11} - l_1 - p_1 + p_1\, l_1 &= 0, \\ l_{12} + p_{12} + l_{11} + p_{11} + p_1\, l_{11} &= 0, \\ l_{21} + p_{21} + l_{11} + p_{11} + p_{11}\, l_1 &= 0, \end{aligned} \tag{2.1}$$

and, for every block $i \geq 2$,

$$\begin{aligned} l_{ii} + p_{ii} + l_{i-1,i} + p_{i-1,i} + p_{i-1,i}\, l_1 &= 0, \\ l'_{ii} + p'_{ii} + l_{i,i-1} + p_{i,i-1} + p_1\, l_{i,i-1} &= 0, \\ l_{i,i+1} + p_{i,i+1} + l_{ii} + p_{ii} + p_1\, l_{ii} &= 0, \\ l_{i+1,i} + p_{i+1,i} + l'_{ii} + p'_{ii} + p'_{ii}\, l_1 &= 0, \end{aligned} \tag{2.2}$$

where for $i = 2$ the symbols $l_{i-1,i}, p_{i-1,i}, l_{i,i-1}, p_{i,i-1}$ stand for $l_{12}, p_{12}, l_{21}, p_{21}$.

The equations (2.1)–(2.2) are triangular: given a point p and a scalar $\lambda \in \mathbb{F}_q$, they determine uniquely the line $N_L(p;\lambda)$ adjacent to p whose first coordinate equals λ, by solving block by block. The map $N_P([l];\mu)$ is defined symmetrically.

3 Constructing Small Cycles

In this section, we prove Theorem 1.1. We start by introducing the involution that drives the construction.

Since $n \geq 2$, we can fix any $\theta \in \mathbb{F}_q \setminus \mathbb{F}_3$ and set

$$\bar{\theta} := -\theta - 1,$$

so that $\theta + \bar{\theta} + 1 = 0$. The map $\lambda \mapsto -\lambda - 1$ on $\mathbb{F}_q$ has a unique fixed point at $\lambda = 1$. In particular, since $\theta \notin \mathbb{F}_3$, we have $\bar{\theta} \neq \theta$ and $\theta, \bar{\theta} \notin \mathbb{F}_3$.

Next, we describe the 10-walk that we want to construct. Setting $u_0 := (0,0,0,\ldots) \in P$, define

$$\begin{aligned} &v_1 = N_L(u_0;\theta), \quad u_2 = N_P(v_1;\,2), \quad v_3 = N_L(u_2;\,2), \\ &u_4 = N_P(v_3;\,\bar{\theta}), \quad v_5 = N_L(u_4;\,0), \quad u_6 = N_P(v_5;\theta), \\ &v_7 = N_L(u_6;\,2), \quad u_8 = N_P(v_7;\,2), \quad v_9 = N_L(u_8;\,\bar{\theta}). \end{aligned} \tag{3.1}$$

By construction, every consecutive pair of vertices in (3.1) is adjacent. To close the walk into a 10-cycle, we need to show that v_9 and u_0 are adjacent, i.e.,

$$v_9 = N_L(u_0;\bar{\theta}). \tag{3.2}$$

We will derive (3.2) by symbolically tracking the walk over the polynomial ring $\mathbb{F}_3[X]$ and showing that the midpoint line v_5 is invariant under the involution $X \mapsto -X - 1$.

3.1 The symbolic walk and the midpoint identity

Let $R := \mathbb{F}_3[X]$ and $Y := -X-1 \in R$. Let $\tau\colon R \to R$ denote the $\mathbb{F}_3$-algebra involution defined by $\tau(X) = Y$. Its fixed subring is $\mathbb{F}_3[C]$, where

$$C := X(X+1) = X^2 + X.$$

Indeed, $\tau(C) = Y(Y+1) = (-X-1)(-X) = X^2 + X = C$, and the extension $R/\mathbb{F}_3[C]$ is a free module of rank 2 on the basis $\{1, X\}$. For $f \in R$, we will write

$$\Delta f := f - \tau f = f(X) - f(Y).$$

Now consider the symbolic 5-step walk

$$u_0 \xrightarrow{X} v_1(X) \xrightarrow{2} u_2(X) \xrightarrow{2} v_3(X) \xrightarrow{Y} u_4(X) \xrightarrow{0} m(X), \tag{3.3}$$

in which the first coordinates of the five intermediate vertices are X, 2, 2, Y and 0. We treat X as a formal variable and view each vertex coordinate as an element of R. Substituting $X = \theta$ recovers v_1, u_2, v_3, u_4 in (3.1) and gives $v_5 = m(\theta)$. Applying τ and then substituting $X = \theta$ yields a parallel 5-walk from u_0 with first-coordinate words $\bar{\theta}, 2, 2, \theta, 0$ that ends at $m(\bar{\theta})$.

Our key claim is the following midpoint identity.

Proposition 3.1. *Every coordinate of $m(X)$ lies in $\mathbb{F}_3[C]$, or equivalently, $m(X) = \tau\big(m(X)\big)$.*

Before proving the proposition, we show that it implies Theorem 1.1.

3.2 Proof of Theorem 1.1 assuming Proposition 3.1

Substituting $X = \theta$ in Proposition 3.1 yields $m(\theta) = m(\bar{\theta})$. Therefore, the two symbolic 5-paths

$$u_0 \xrightarrow{\theta}\xrightarrow{2}\xrightarrow{2}\xrightarrow{\bar{\theta}}\xrightarrow{0} m \qquad \text{and} \qquad u_0 \xrightarrow{\bar{\theta}}\xrightarrow{2}\xrightarrow{2}\xrightarrow{\theta}\xrightarrow{0} m$$

end at the same line $m = v_5$. Reading the second path in reverse from v_5, and using the fact that neighbors with a prescribed first coordinate are unique, we see that its successive vertices are u_6, v_7, u_8, v_9 from (3.1). The initial edge $\{u_0, N_L(u_0; \bar{\theta})\}$ of the second path therefore equals the edge $\{u_0, v_9\}$, proving (3.2).

It remains to verify that the ten vertices of the closed walk are pairwise distinct. Restricting to the first two coordinates of each vertex, our computations in Section 3.3 below (specifically, equations (3.4)–(3.7)) give the following table:

point	$(\cdot)_1, (\cdot)_{11}$	line	$(\cdot)_1, (\cdot)_{11}$
u_0	$(0, 0)$	v_1	$(\theta,\ \theta)$
u_2	$(2,\ \theta - 1)$	v_3	$(2,\ 1 - \theta)$
u_4	$(\bar{\theta},\ \bar{\theta})$	v_5	$(0, 0)$
u_6	$(\theta,\ \theta)$	v_7	$(2,\ \theta - 1)$
u_8	$(2,\ 1 - \theta)$	v_9	$(\bar{\theta},\ \bar{\theta})$

Here we have used $(u_2)_{11} = X + 2 = \theta - 1$ and $(v_3)_{11} = 1 + 2X = 1 - \theta$ in characteristic 3 after substituting $X = \theta$. The second half of the table follows by τ-symmetry. Since $\theta \notin \mathbb{F}_3$, we have $\theta \neq \bar{\theta}$ (else $2\theta = -1 = 2$, so $\theta = 1 \in \mathbb{F}_3$), $\theta \neq 0, 2$ and $\bar{\theta} \neq 0, 2$. Moreover, $\theta - 1 \neq 1 - \theta$ (else $\theta = 1 \in \mathbb{F}_3$). Therefore, the five point-pairs are pairwise distinct, and so are the five line-pairs. Hence the closed walk is a simple cycle of length 10.

Finally, for every $k \geq 2$, taking the first k coordinates of each vertex in the displayed walk gives a closed 10-walk in $D'(k, q)$ satisfying the corresponding defining relations. The first two coordinates already distinguish the vertices on each side, so the walk remains a C_{10}. This completes the proof of Theorem 1.1, modulo Proposition 3.1. □

3.3 Proof of Proposition 3.1

We compute the symbolic walk (3.3) block by block.

From $u_0 = 0$ and (2.1), taking $(v_1)_1 = X$ gives

$$(v_1)_1 = X, \quad (v_1)_{11} = X, \quad (v_1)_{12} = -X, \quad (v_1)_{21} = -X. \tag{3.4}$$

Next, $u_2 = N_P(v_1; 2)$ has $(u_2)_1 = 2$. The first equation in (2.1) gives $(u_2)_{11} = (v_1)_1 + (u_2)_1 - (v_1)_{11} - (u_2)_1(v_1)_1 = X + 2 - X - 2X = 2 - 2X = X + 2$ in $\mathbb{F}_3$. Solving the rest of (2.1) yields

$$(u_2)_1 = 2, \quad (u_2)_{11} = X + 2, \quad (u_2)_{12} = 1, \quad (u_2)_{21} = 1 + 2X^2. \tag{3.5}$$

For $v_3 = N_L(u_2; 2)$ and $u_4 = N_P(v_3; Y)$, similar computations using (2.1) give

$$(v_3)_1 = 2, \quad (v_3)_{11} = 1 + 2X, \quad (v_3)_{12} = -X, \quad (v_3)_{21} = 1 + X + X^2, \tag{3.6}$$

$$(u_4)_1 = Y, \quad (u_4)_{11} = Y, \quad (u_4)_{12} = 1 - X^2, \quad (u_4)_{21} = 1 - X^2. \tag{3.7}$$

For the final move $u_4 \to m$ we have $l_1 = m_1 = 0$, which annihilates every product involving l_1 in (2.1)–(2.2). Therefore at levels 0 and 1,

$$\begin{aligned}
m_1 &= 0, \\
m_{11} &= -\big[(u_4)_{11} - (u_4)_1\big] = -(Y - Y) = 0, \\
m_{12} &= -\big[(u_4)_{12} + (u_4)_{11} + m_{11}\big] = -\big[(1 - X^2) + Y\big] = X^2 + X = C, \\
m_{21} &= -\big[(u_4)_{21} + (u_4)_{11}\big] = -\big[(1 - X^2) + Y\big] = X^2 + X = C,
\end{aligned}$$

using $-1 - Y = -1 - (-X - 1) = X$ in $\mathbb{F}_3$. Similarly, an explicit computation at block 2 gives

$$m_{22} = 0, \quad m'_{22} = 0, \quad m_{23} = m_{32} = C, \tag{3.8}$$

all in $\mathbb{F}_3[C]$.

It now remains to verify that for every block $i \geq 3$, all four coordinates of m at block i lie in $\mathbb{F}_3[C]$. We will prove this by induction on i.

The inductive step

Suppose that all coordinates of m at blocks $\leq i-1$ lie in $\mathbb{F}_3[C]$, i.e., $\Delta m_* = 0$ at those blocks. We want to show the same at block i. Applying (2.2) to the move $u_4 \to m$, with $l_1 = m_1 = 0$, gives

$$\begin{aligned} m_{ii} &= -\big(u_{4,\,ii} + m_{i-1,i} + u_{4,\,i-1,i}\big), \\ m'_{ii} &= -\big(u'_{4,\,ii} + u_{4,\,i,i-1} + (1+Y)\,m_{i,i-1}\big), \\ m_{i,i+1} &= -\big(u_{4,\,i,i+1} + u_{4,\,ii} + (1+Y)\,m_{ii}\big), \\ m_{i+1,i} &= -\big(u_{4,\,i+1,i} + u'_{4,\,ii} + m'_{ii}\big). \end{aligned} \tag{3.9}$$

The first and fourth lines have no $(1+Y)$ factor because $l_1 = m_1 = 0$. The second and third lines retain the factor $1+Y = -X$ from the grouped terms $l_* + p_1\,l_* = (1+p_1)\,l_*$ with $p_1 = (u_4)_1 = Y$.

Applying Δ to (3.9) and using the inductive hypothesis $\Delta m_{i-1,i} = \Delta m_{i,i-1} = 0$, we obtain

$$\begin{aligned} \Delta m_{ii} &= -\Delta u_{4,\,ii} - \Delta u_{4,\,i-1,i}, \\ \Delta m'_{ii} &= -\Delta u'_{4,\,ii} - \Delta u_{4,\,i,i-1} - (Y-X)\,m_{i,i-1}, \\ \Delta m_{i,i+1} &= -\Delta u_{4,\,i,i+1} - \Delta u_{4,\,ii} - (Y-X)\,m_{ii}, \\ \Delta m_{i+1,i} &= -\Delta u_{4,\,i+1,i} - \Delta u'_{4,\,ii} - \Delta m'_{ii}. \end{aligned} \tag{3.10}$$

The second line uses $\Delta((1+Y)\,m_{i,i-1}) = \Delta(1+Y)\cdot m_{i,i-1} = (Y-X)\,m_{i,i-1}$, which holds because $m_{i,i-1}$ is τ-fixed by the inductive hypothesis. The third line is analogous, once we know $\Delta m_{ii} = 0$ from the first line.

We see from (3.10) that $\Delta m_* = 0$ holds at block i if the following four identities hold for u_4 at block i:

$$\Delta u_{4,\,ii} + \Delta u_{4,\,i-1,i} = 0, \tag{A}$$

$$\Delta u'_{4,\,ii} + \Delta u_{4,\,i,i-1} + (Y-X)\,m_{i,i-1} = 0, \tag{B}$$

$$\Delta u_{4,\,i,i+1} + \Delta u_{4,\,ii} + (Y-X)\,m_{ii} = 0, \tag{C}$$

$$\Delta u_{4,\,i+1,i} + \Delta u'_{4,\,ii} = 0. \tag{D}$$

Indeed, from (A) and the first line of (3.10), $\Delta m_{ii} = 0$. Then (C) and the third line of (3.10) give $\Delta m_{i,i+1} = 0$. Similarly, (B) gives $\Delta m'_{ii} = 0$, and then (D) gives $\Delta m_{i+1,i} = 0$.

It therefore suffices to verify (A)–(D) at every block $i \geq 2$. We do this in the next claim.

Claim 3.2. The identities (A)–(D) hold at every block $i \geq 2$.

The proof of Claim 3.2 is a direct but somewhat long calculation from the triangular defining equations. We give the details in Appendix A. A computational verification was also performed until $k \leq 24$, whose details are available in the repository [4].

Claim 3.2, together with the implication (A)–(D) $\Rightarrow \Delta m_* = 0$ derived from (3.10), shows that $\Delta m_* = 0$ at every block $i \geq 1$. This finishes the proof of Proposition 3.1. $\square$

Remark 3.3. The hypothesis $\theta \notin \mathbb{F}_3$ is essential: the simplicity check fails when $\theta \in \mathbb{F}_3$. Computationally, $D'(k,3)$ has girth at least 12 for every $k \leq 18$.

Remark 3.4. For even powers $q = 3^{2r}$, one can also use the order-2 Frobenius $\sigma\colon x \mapsto x^{3^r}$ to construct the 10-cycle by choosing θ with $\sigma(\theta) = \bar{\theta}$. The proof given here is uniform across all $n \geq 2$.

4 Concluding Remarks

Theorem 1.1 shows that $D'(k,q)$ has girth at most 10 for every $q = 3^n$ with $n \geq 2$ and every $k \geq 5$. This rules out using $D_3(k,q)$ as a uniform construction for lower bounds on $\mathrm{ex}_3(n, \widetilde{C}_{2k})$ beyond $k = 4$.

The proof presented in this note was motivated by computations. A representative 10-cycle arising from the Frobenius automorphism was first found empirically for the graphs $D'(k,4)$, and this pattern was later extended to other finite field extensions. For completeness, we include the original proof for $D'(k,4)$ in Appendix B.

The case $q = 3$ remains exceptional. Computations in the `dgraphs` repository show that $D'(k,3)$ matches $D(k,3)$ in girth and component count for all $k \leq 18$, even though $D'(11,3) \not\cong D(11,3)$. This suggests a genuine structural coincidence in the prime field case, but Theorem 1.1 shows that such behavior does not carry over to nontrivial extensions of $\mathbb{F}_3$.

We also note that the component structure of $D'(k,q)$ appears to contain some further arithmetic features. For instance, the computations for $q = 4$ show anomalous component counts compared with the corresponding data for $D(k,4)$ and for $D'(k,q)$ at other small values of q. While some of these computations are available in the repository [4], we leave this avenue as future work.

Future constructions aimed at lower bounds on $\mathrm{ex}_3(n, \widetilde{C}_{2k})$ for $k \geq 5$ should therefore look for new link families, or for modifications of $D_3(k,q)$, rather than attempting to use $D'(k,q)$ over nontrivial extensions of $\mathbb{F}_3$.

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A Proof of Claim 3.2

We prove Claim 3.2 by a direct calculation from the triangular defining equations of $D'(k,q)$, keeping the notation from the proof of Proposition 3.1. We work in the ring $R = \mathbb{F}_3[X,Y]/(X+Y+1)$, where $\tau(X) = Y$, $\tau(Y) = X$ and $\Delta f = f - \tau(f)$. In particular, $1+Y = -X$ and $\Delta(1+Y) = Y - X$.

Write the block-i coordinates of u_4 as

$$a_i = u_{4,ii}, \quad b_i = u'_{4,ii}, \quad c_i = u_{4,i,i+1}, \quad d_i = u_{4,i+1,i},$$

and the block-i coordinates of the midpoint line m as

$$A_i = m_{ii}, \quad B_i = m'_{ii}, \quad C_i = m_{i,i+1}, \quad D_i = m_{i+1,i}.$$

The four auxiliary identities to be verified are

$$\begin{aligned} \Delta a_i + \Delta c_{i-1} &= 0, \\ \Delta b_i + \Delta d_{i-1} + (Y-X)\,D_{i-1} &= 0, \\ \Delta c_i + \Delta a_i + (Y-X)\,A_i &= 0, \\ \Delta d_i + \Delta b_i &= 0. \end{aligned} \tag{A.1}$$

Define two polynomial sequences by

$$P_2 = Y, \quad P_{i+1} = (1+X)(P_i - X) \quad (i \geq 2),$$

and

$$S_1 = 1 - X^2, \quad S_{i+1} = (1+X)(X + S_i) \quad (i \geq 1).$$

Solving the triangular equations for the first part of the walk $u_0 \to v_1 \to u_2 \to v_3$ gives, for every $i \geq 2$,

$$\begin{aligned} (v_3)_{ii} &= X - P_i, & (v_3)'_{ii} &= X, \\ (v_3)_{i,i+1} &= -X, & (v_3)_{i+1,i} &= -X - S_i. \end{aligned} \tag{A.2}$$

The identity $(v_3)'_{ii} = X$ follows because the moves $v_1 \to u_2$ and $u_2 \to v_3$ both have first coordinates 2, so the factors $1+2=0$ in characteristic 3 cancel the cross-terms.

For the move $v_3 \to u_4$, the first coordinate of u_4 is Y and the first coordinate of v_3 is 2. Applying (2.2) with $p_1 = Y$ and $l_1 = 2$ gives

$$\begin{aligned} a_i &= -\big((v_3)_{ii} + (v_3)_{i-1,i}\big), \\ b_i &= -\big((v_3)'_{ii} + d_{i-1} + (1+Y)(v_3)_{i,i-1}\big), \\ c_i &= -\big((v_3)_{i,i+1} + a_i + (1+Y)(v_3)_{ii}\big), \\ d_i &= -\big((v_3)_{i+1,i} + (v_3)'_{ii}\big). \end{aligned} \tag{A.3}$$

Substituting (A.2) and using $1+Y=-X$, the first and third lines give

$$a_i = P_i, \quad c_i = -P_{i+1} \quad (i \geq 2).$$

The fourth line gives $d_i = S_i$ for $i \geq 2$. The second line gives $b_i = -S_i$ for $i \geq 3$; at $i = 2$ the formula involves the initial coordinate $d_1 = u_{4,21}$ from block 1.

From the move $u_4 \to m$ (with $l_1 = m_1 = 0$), the recursions for the midpoint are

$$A_i = -(a_i + C_{i-1} + c_{i-1}), \quad C_i = -(c_i + a_i + (1 + Y)\, A_i).$$

Since $a_i + c_{i-1} = P_i + (-P_i) = 0$ for $i \geq 3$, the first equation simplifies to $A_i = -C_{i-1}$. Using $1 + Y = -X$, the second gives $C_i = P_{i+1} - P_i + X\, A_i$. Combining,

$$A_i = P_{i-1} - P_i - X\, A_{i-1} \quad (i \geq 3), \quad A_2 = 0. \tag{A.4}$$

Similarly, using $d_i = S_i$ and $b_i = -S_i$ for $i \geq 3$, the midpoint equations give

$$D_i = S_{i-1} - S_i - X\, D_{i-1} \quad (i \geq 3), \quad D_2 = X^2 + X. \tag{A.5}$$

We now prove two key identities by induction.

Claim A.1 (First identity)**.**

$$(Y - X)\, A_i = \Delta P_{i+1} - \Delta P_i \quad (i \geq 2). \tag{A.6}$$

Proof. For $i = 2$ this is immediate since $A_2 = 0$ and $\Delta P_3 - \Delta P_2 = 0$.

For the inductive step, set $p = P_{i-1}$ and $q = \tau(P_{i-1})$, so that $P_i = (1 + X)(p - X)$ and $\tau(P_i) = (1 + Y)(q - Y)$. Expanding

$$(\Delta P_{i+1} - \Delta P_i) + X(\Delta P_i - \Delta P_{i-1}) - (Y - X)(P_{i-1} - P_i)$$

and collecting terms gives $(X + Y + 1)(-X^2 + Xp - X + Y^2 - Yq + Y)$, which vanishes because $X + Y + 1 = 0$ in R. Therefore the expression $(\Delta P_{i+1} - \Delta P_i)/(Y - X)$ satisfies the same recurrence (A.4) and the same initial value as A_i, proving (A.6)[3]. □

Claim A.2 (Second identity)**.**

$$(Y - X)\, D_i = \Delta S_{i+1} - \Delta S_i \quad (i \geq 2). \tag{A.7}$$

Proof. The base case $i = 2$ is $(Y - X)(X^2 + X) = \Delta S_3 - \Delta S_2$, which one checks directly. The inductive step is identical: setting $s = S_{i-1}$ and $t = \tau(S_{i-1})$, one expands

$$(\Delta S_{i+1} - \Delta S_i) + X(\Delta S_i - \Delta S_{i-1}) - (Y - X)(S_{i-1} - S_i)$$

and obtains $(X + Y + 1)(X^2 + Xs + X - Y^2 - Yt - Y) = 0$. □

[3]The quotient is a polynomial because each difference $\Delta f = f(X) - f(Y)$ is divisible by $Y - X$.

With (A.6) and (A.7) in hand, the four identities (A.1) follow immediately for $i \geq 3$.

Identity (A): $\Delta a_i + \Delta c_{i-1} = \Delta P_i + \Delta(-P_i) = 0$.

Identity (D): $\Delta d_i + \Delta b_i = \Delta S_i + \Delta(-S_i) = 0$.

For identity (C),

$$\Delta c_i + \Delta a_i + (Y - X)\, A_i = -\Delta P_{i+1} + \Delta P_i + (\Delta P_{i+1} - \Delta P_i) = 0.$$

For identity (B),

$$\Delta b_i + \Delta d_{i-1} + (Y - X)\, D_{i-1} = -\Delta S_i + \Delta S_{i-1} + (\Delta S_i - \Delta S_{i-1}) = 0.$$

It remains to verify (A.1) at $i = 2$. The relevant values are

$$a_2 = Y, \quad c_1 = d_1 = c_2 = 1 - X^2, \quad b_2 = X^3 - X^2 - 1, \quad d_2 = -X^3 - X + 1,$$

and $A_2 = 0$, $m_{21} = X^2 + X$. Since $Y - X = X - 1$ in $\mathbb{F}_3[X]$,

$$\begin{aligned} \Delta a_2 + \Delta c_1 &= (X - 1) + (1 - X) = 0, \\ \Delta b_2 + \Delta d_1 + (Y - X)\, m_{21} &= (-X^3 - X - 1) + (1 - X) + (X^3 - X) = 0, \\ \Delta c_2 + \Delta a_2 + (Y - X)\, A_2 &= (1 - X) + (X - 1) + 0 = 0, \\ \Delta d_2 + \Delta b_2 &= (X^3 + X + 1) + (-X^3 - X - 1) = 0. \end{aligned}$$

This completes the proof of Claim 3.2.

B Motivating calculation on the 4-element finite field

This appendix records the proof that $D'(k, 4)$ contains a 10-cycle. This argument originally motivated the search for short cycles arising from field symmetries. It works only in characteristic 2, and is not used in the proof of Theorem 1.1. The main proof uses a different involution and works in characteristic 3.

Theorem B.1. *For every $k \geq 2$, the graph $D'(k, 4)$ contains a cycle of length* 10.

Proof. Let $\mathbb{F}_4 = \mathbb{F}_2[\alpha]/(\alpha^2 + \alpha + 1)$, so that $\alpha^2 = \alpha + 1$ and $1 + \alpha + \alpha^2 = 0$. Since $\mathrm{char}(\mathbb{F}_4) = 2$, we may read every minus sign in the defining relations (2.1)–(2.2) as a plus sign.

We will define five vertices $u_0, u_2, u_4 \in P$ and $v_1, v_3, v_5 \in L$ such that

$$u_0 \sim v_1 \sim u_2 \sim v_3 \sim u_4 \sim v_5.$$

For this purpose, define for $i \geq 1$ the values

$$r_i = \begin{cases} 0, & i \equiv 2 \pmod 3, \\ 1, & i \equiv 0 \pmod 3, \\ \alpha, & i \equiv 1 \pmod 3, \end{cases} \qquad t_i = \begin{cases} 0, & i \equiv 2 \pmod 3, \\ 1, & i \equiv 0, 1 \pmod 3. \end{cases}$$

Then r_i satisfy the recurrences $r_{i-1} + \alpha r_i + \alpha = 0$ and $r_i + \alpha r_{i+1} + \alpha = 0$ for every $i \geq 2$.

The five vertices are given as follows. First, $u_0 = (0, 0, \ldots)$ and $v_1 = (\alpha, \alpha, \alpha, \alpha, \ldots)$. Next, $u_2 \in P$ is defined by

$$(u_2)_1 = 1, \quad (u_2)_{11} = \alpha + 1, \quad (u_2)_{12} = 1, \quad (u_2)_{21} = \alpha,$$

and for $i \geq 2$,

$$(u_2)_{ii} = \alpha + 1, \quad (u_2)'_{ii} = \alpha r_i, \quad (u_2)_{i,i+1} = 1, \quad (u_2)_{i+1,i} = r_i.$$

The line $v_3 \in L$ is given by

$$(v_3)_1 = 1, \quad (v_3)_{11} = \alpha, \quad (v_3)_{12} = \alpha, \quad (v_3)_{21} = 0,$$

and for $i \geq 2$,

$$(v_3)_{ii} = 1, \quad (v_3)'_{ii} = \alpha, \quad (v_3)_{i,i+1} = \alpha, \quad (v_3)_{i+1,i} = \alpha r_{i+1}.$$

The point $u_4 \in P$ is given by

$$(u_4)_1 = \alpha + 1, \quad (u_4)_{11} = \alpha + 1, \quad (u_4)_{12} = \alpha, \quad (u_4)_{21} = \alpha,$$

and for $i \geq 2$,

$$(u_4)_{ii} = \alpha + 1, \quad (u_4)'_{ii} = r_i, \quad (u_4)_{i,i+1} = \alpha + 1, \quad (u_4)_{i+1,i} = r_i.$$

Finally, the line $v_5 \in L$ is given by

$$(v_5)_1 = 0, \quad (v_5)_{11} = 0, \quad (v_5)_{12} = 1, \quad (v_5)_{21} = 1,$$

and for $i \geq 2$,

$$(v_5)_{ii} = 0, \quad (v_5)'_{ii} = t_i, \quad (v_5)_{i,i+1} = 0, \quad (v_5)_{i+1,i} = t_i.$$

A direct verification of the adjacencies $u_0 \sim v_1 \sim u_2 \sim v_3 \sim u_4 \sim v_5$ follows by substituting these formulas into (2.1)–(2.2) and using the recurrences for r_i, together with the characteristic-2 simplification $1 + \alpha + \alpha^2 = 0$.

To close the walk, we apply the Frobenius automorphism $\sigma \colon \mathbb{F}_4 \to \mathbb{F}_4$ defined by $\sigma(x) = x^2$. Since all coefficients in (2.1)–(2.2) lie in $\mathbb{F}_2$, applying σ coordinatewise gives an automorphism of $D'(k, 4)$. We set

$$u_6 = \sigma(u_4), \quad v_7 = \sigma(v_3), \quad u_8 = \sigma(u_2), \quad v_9 = \sigma(v_1).$$

Note that $\sigma(u_0) = u_0$ trivially, and $\sigma(v_5) = v_5$ because every coordinate of v_5 already lies in $\mathbb{F}_2$. Therefore, we obtain the further adjacencies $u_6 \sim v_5$, $u_6 \sim v_7$, $u_8 \sim v_7$, $u_8 \sim v_9$ and $u_0 \sim v_9$. Consequently, the ten vertices $u_0, v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_8, v_9$ form a closed walk of length 10.

It remains to check pairwise distinctness, which follows from inspecting the first two coordinates of each vertex:

$$u_0 = (0, 0, \ldots),\ u_2 = (1, \alpha + 1, \ldots),\ u_4 = (\alpha + 1, \alpha + 1, \ldots),\ u_6 = (\alpha, \alpha, \ldots),\ u_8 = (1, \alpha, \ldots),$$

and

$$v_1 = (\alpha, \alpha, \ldots),\ v_3 = (1, \alpha, \ldots),\ v_5 = (0, 0, \ldots),\ v_7 = (1, \alpha + 1, \ldots),\ v_9 = (\alpha + 1, \alpha + 1, \ldots).$$

Truncating to the first k coordinates gives the same C_{10} in $D'(k, 4)$ for every $k \geq 2$. □

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